

Relativistic Quantum Clocks

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Based on work with Mehdi Ahmadi, Philipp A. Höhn, and Maximilian P. Lock



The Time Machine Factory, 2019

A. R. H. Smith and M. Ahmadi, *Quantizing time: Interacting clocks and systems*, Quantum 3, 160 (2019)

A. R. H. Smith and M. Ahmadi, *Relativistic quantum clocks observe classical and quantum time dilation*,
arXiv:1904.12390 [quant-ph] (2019)

P. A. Höhn, M. P. E. Lock, and A. R. H. Smith, *The trinity of relational quantum dynamics*, Forthcoming (2019)

What is time?

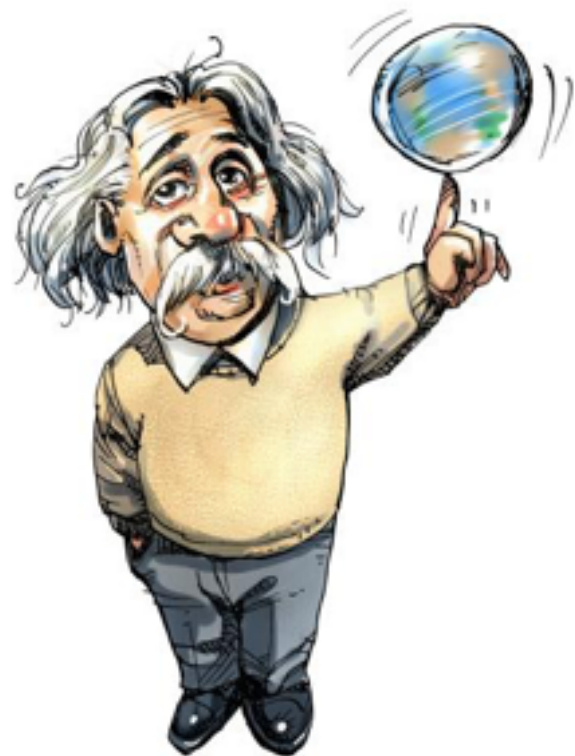


Newtonian mechanics

$$F = ma \quad \Leftrightarrow \quad \frac{dp}{dt} = m \frac{d^2 x}{dt^2}$$

Quantum mechanics

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle$$



General relativity

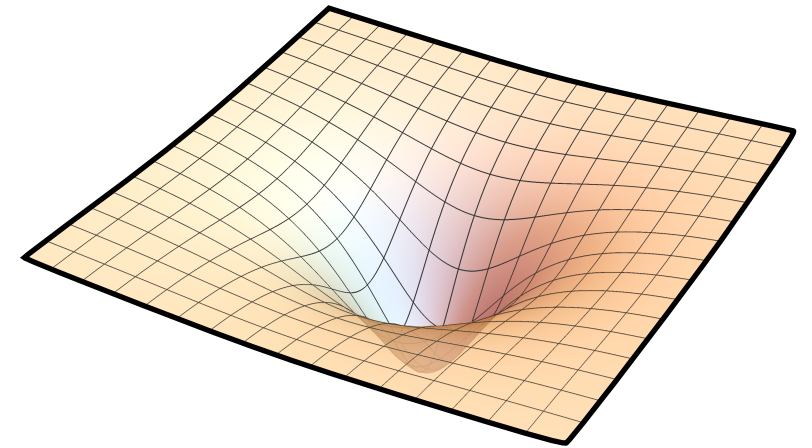
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$



Quantum gravity and the problem of time

We need to choose canonical variables:

Choose the 3-metric γ_{ab} and its
conjugate momentum P^{ab}



Then we can express general relativity in the Hamiltonian form

$$H_{\text{GR}}[\gamma, P] = 16\pi G G_{abcd}[\gamma] P^{ab} P^{cd} + V[\gamma] + \sqrt{\gamma} \rho \approx 0$$

and three momentum constraints

1. Constraint

$$H_{\text{GR}} |\psi\rangle\rangle = 0$$

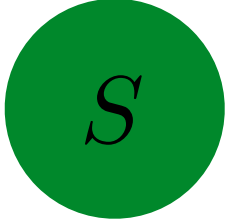
2. Dynamics

$$i \frac{d}{dt} |\psi\rangle\rangle = H_{\text{GR}} |\psi\rangle\rangle = 0$$

} $|\psi\rangle\rangle$ is frozen!



Time as a dynamical variable

System (q, q')  with action $\mathcal{S} = \int_{t_1}^{t_2} dt L_S (q, q')$

Introducing an arbitrary integration parameter τ , the action is

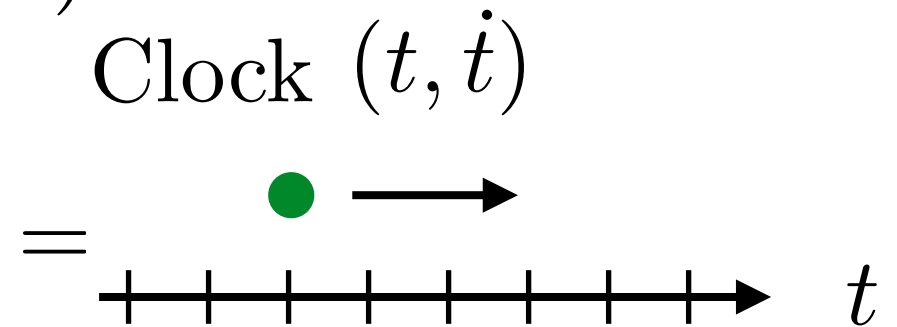
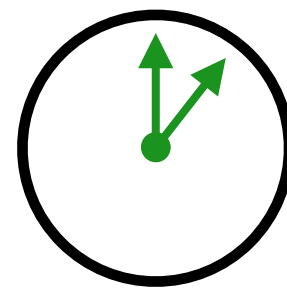
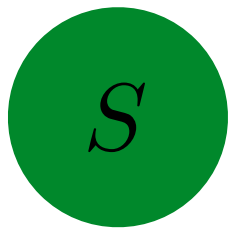
$$\mathcal{S} = \int_{\tau_1}^{\tau_2} d\tau \dot{t} L_S (q, \dot{q}/\dot{t}) = \int_{\tau_1}^{\tau_2} d\tau L (q, \dot{q}, \dot{t})$$

$$\dot{t} = \frac{dt}{d\tau}$$

where

$$L (q, \dot{q}, \dot{t}) := \dot{t} L_S (q, \dot{q}/\dot{t})$$

System (q, \dot{q})



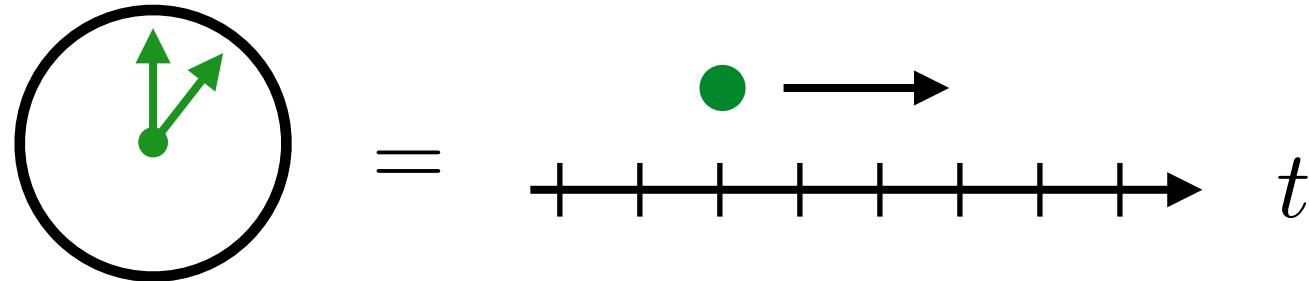
The total (super) Hamiltonian is constrained to vanish

$$H = P_t + H_S \approx 0$$

How do we recover time evolution?

$$H |\psi\rangle\rangle = (P_t \otimes I_S + I_C \otimes H_S) |\psi\rangle\rangle = 0$$

Different times are indicated by different states of the clock



Clock state indicating time t :

$$|t\rangle := e^{-iP_t t} |t_0\rangle \quad \langle t_2 | t_1 \rangle = \delta(t_2 - t_1) \quad I_C = \int dt |t\rangle \langle t|$$

The state of the system at the time t is the joint state of the clock and system $|\psi\rangle\rangle$ conditioned on the clock being in the state $|t\rangle$:

$$S \quad |\psi_S(t)\rangle := \langle t | \otimes I_S |\psi\rangle\rangle$$

$$i \frac{d}{dt} |\psi_S(t)\rangle = H_S |\psi_S(t)\rangle$$

Is this the most general total Hamiltonian?

$$H |\psi\rangle\rangle = (P_t \otimes I_S + I_C \otimes H_S) |\psi\rangle\rangle = 0$$

1. We can consider a general clock Hamiltonian: $P_t \rightarrow H_C$
2. We can include an interaction Hamiltonian: H_{int}

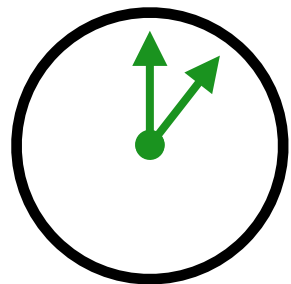
$$H_{\text{GR}}[\gamma, P] = 16\pi G G_{abcd}[\gamma] P^{ab} P^{cd} + V[\gamma] + \underline{\sqrt{\gamma}\rho} \approx 0$$

General total Hamiltonian

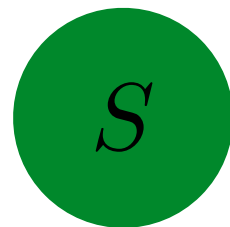
$$H |\psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S + \lambda H_{int}) |\psi\rangle\rangle = 0$$

The modified Schrödinger equation

$$H |\psi\rangle\rangle = (H_C \otimes I_S + I_C \otimes H_S + \lambda H_{int}) |\psi\rangle\rangle = 0$$



$$|t\rangle := e^{-iH_C t} |t_0\rangle$$



$$|\psi_S(t)\rangle := (\langle t| \otimes I_S) |\psi\rangle\rangle$$

$$i \frac{d}{dt} |\psi_S(t)\rangle = H_S |\psi_S(t)\rangle + \lambda \int dt' K(t, t') |\psi_S(t')\rangle$$

1. Time non-local
2. Perturbative solution exists

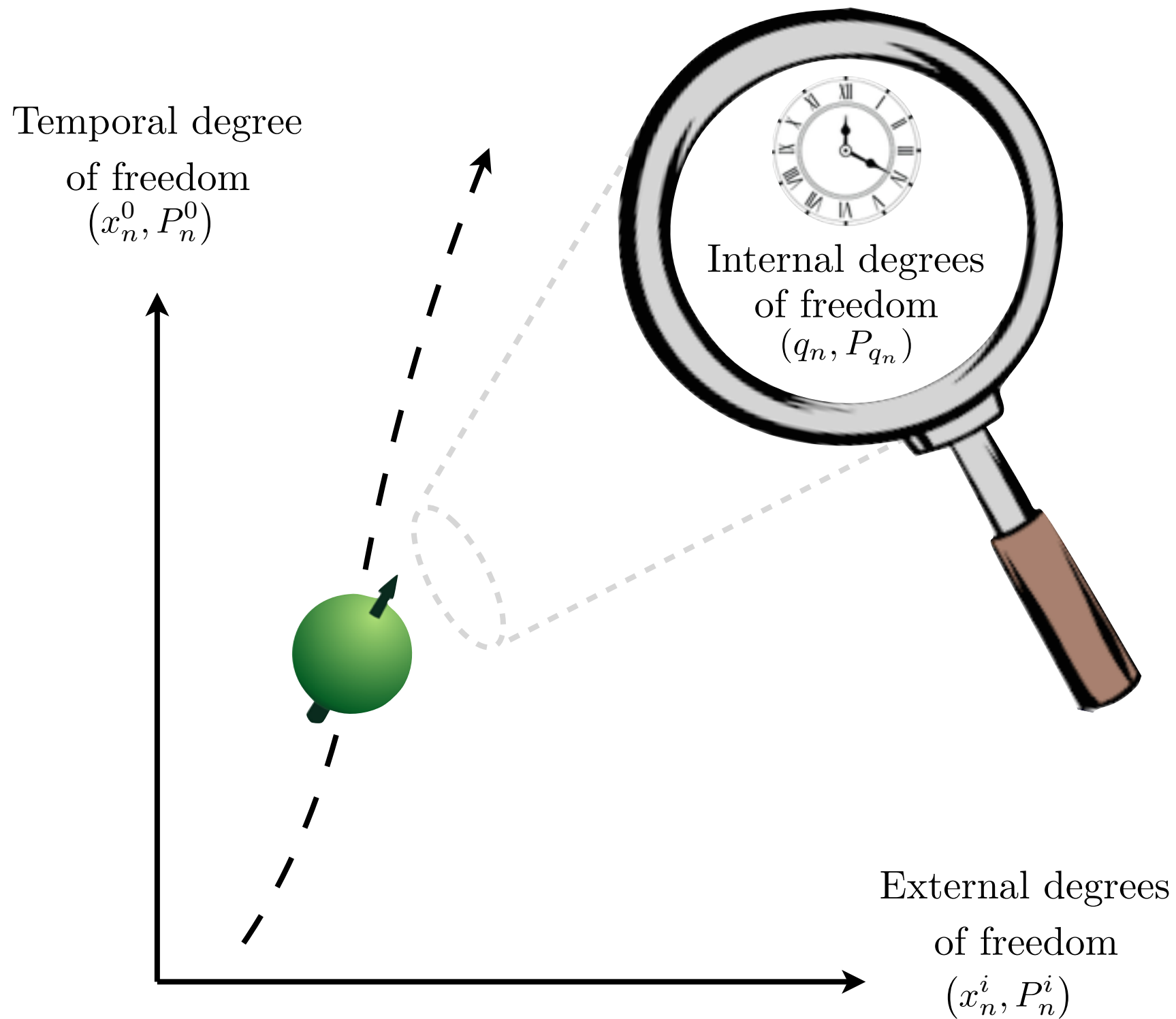
$$K(t, t') := \langle t| H_{int} |t'\rangle$$

Example: Clocks coupled through Newtonian gravity

$$H_{int} = -\frac{G m_C m_S}{d} = -\frac{G}{d} \frac{H_C}{c^2} \otimes \frac{H_S}{c^2}$$

$$i \frac{d}{dt} |\psi_S(t)\rangle = \left[H_S + \frac{G}{c^4 d} H_S^2 + \mathcal{O} \left(\left[\frac{G}{c^4 d} \right]^2 \right) \right] |\psi_S(t)\rangle$$

Relativistic particles as quantum clocks



Starting with an action

$$\begin{aligned}\mathcal{S} &= \int d\tau \left[-mc^2 + P_q \frac{dq}{d\tau} - H^{\text{clock}} \right] \\ &= \int dt \sqrt{-\dot{x}^2} \left(-mc^2 + \frac{P_q \dot{q}}{\sqrt{-\dot{x}^2}} - H^{\text{clock}} \right)\end{aligned}$$

$\dot{x}^2 := g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$

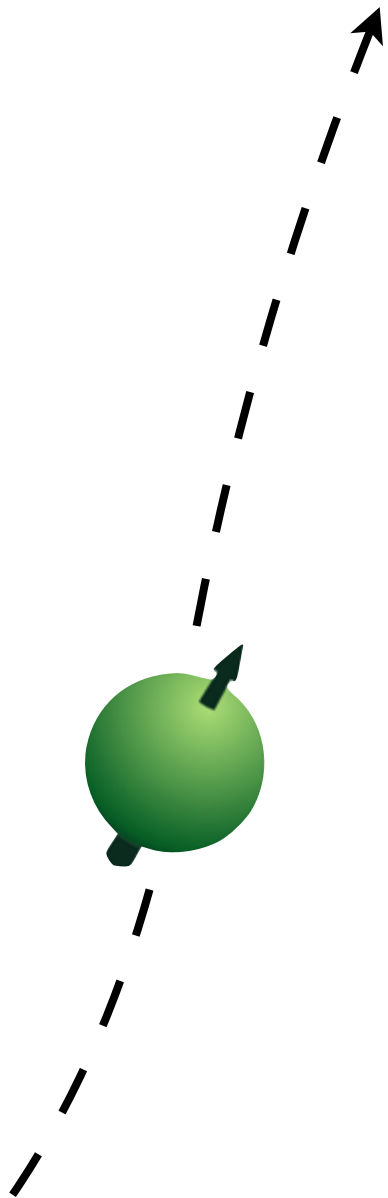
Construct the associated Hamiltonian

$$H = g_{\mu\nu} P^\mu P^\nu + (mc^2 + H^{\text{clock}})^2 \approx 0$$

The Hamiltonian may be factorized as

$$H = C^+ C^- \quad \text{where} \quad C^\pm = P^0 + H_S^\pm \approx 0$$

$$H_S^\pm := \frac{\pm 1}{\sqrt{-g_{00}}} \sqrt{g_{ij} P^i P^j + (mc^2 + H^{\text{clock}})^2}$$

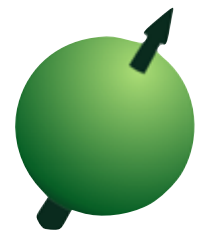


Quantization and recovery of relativistic QM

Quantization

$$C^\pm = P^0 + H_S^\pm \approx 0 \quad \xrightarrow{\text{Quantization}} \quad C^\pm |\psi\rangle\rangle = (P^0 + H_S^\pm) |\psi\rangle\rangle = 0$$

Define the conditional state in the same way as before



$$|\psi_S(t)\rangle := \langle t | \otimes I_S |\psi\rangle\rangle \in \mathcal{H}^{\text{ext}} \otimes \mathcal{H}^{\text{clock}}$$

which satisfies the relativistic Schrödinger equation

$$i \frac{d}{dt} |\psi_S(t)\rangle = H_S^\pm |\psi_S(t)\rangle$$

$$H_S^\pm := \frac{\pm 1}{\sqrt{-g_{00}}} \sqrt{g_{ij} P^i P^j + (mc^2 + H^{\text{clock}})^2}$$

A proper time observable on the clock Hilbert space

$$\rho_C(\tau) = e^{-iH^{\text{clock}}\tau} \rho_C e^{iH^{\text{clock}}\tau}$$

A time observable should satisfy the following properties:

1. On average T_C should estimate the proper time τ
2. The variance in a measurement of T_C should be independent of the proper time τ

These two properties are satisfied if T_C is a covariant POVM

$$E(\tau) = e^{-iH^{\text{clock}}\tau} E(0) e^{iH^{\text{clock}}\tau}$$

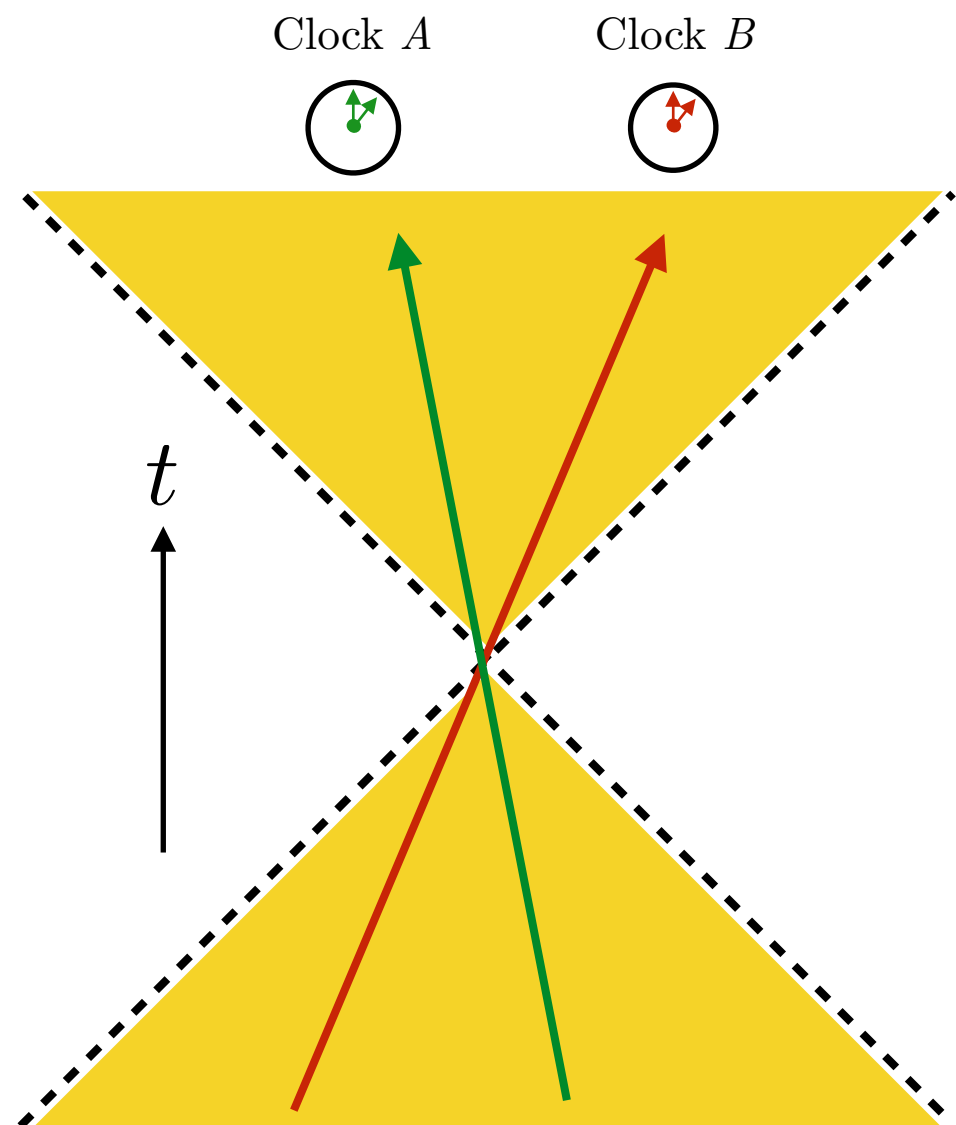
Leads to a proper time-clock energy uncertainty relation

$$\langle \Delta T_C^2 \rangle_{\rho_C} \geq \frac{1}{4 \langle (\Delta H^{\text{clock}})^2 \rangle_{\rho_C}}$$

A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, (1982)

C. W. Helstrom, *Quantum detection and estimation theory*, (1976)

Probabilistic time dilation



$$\text{Prob} [T_A = \tau_A \mid T_B = \tau_B]$$

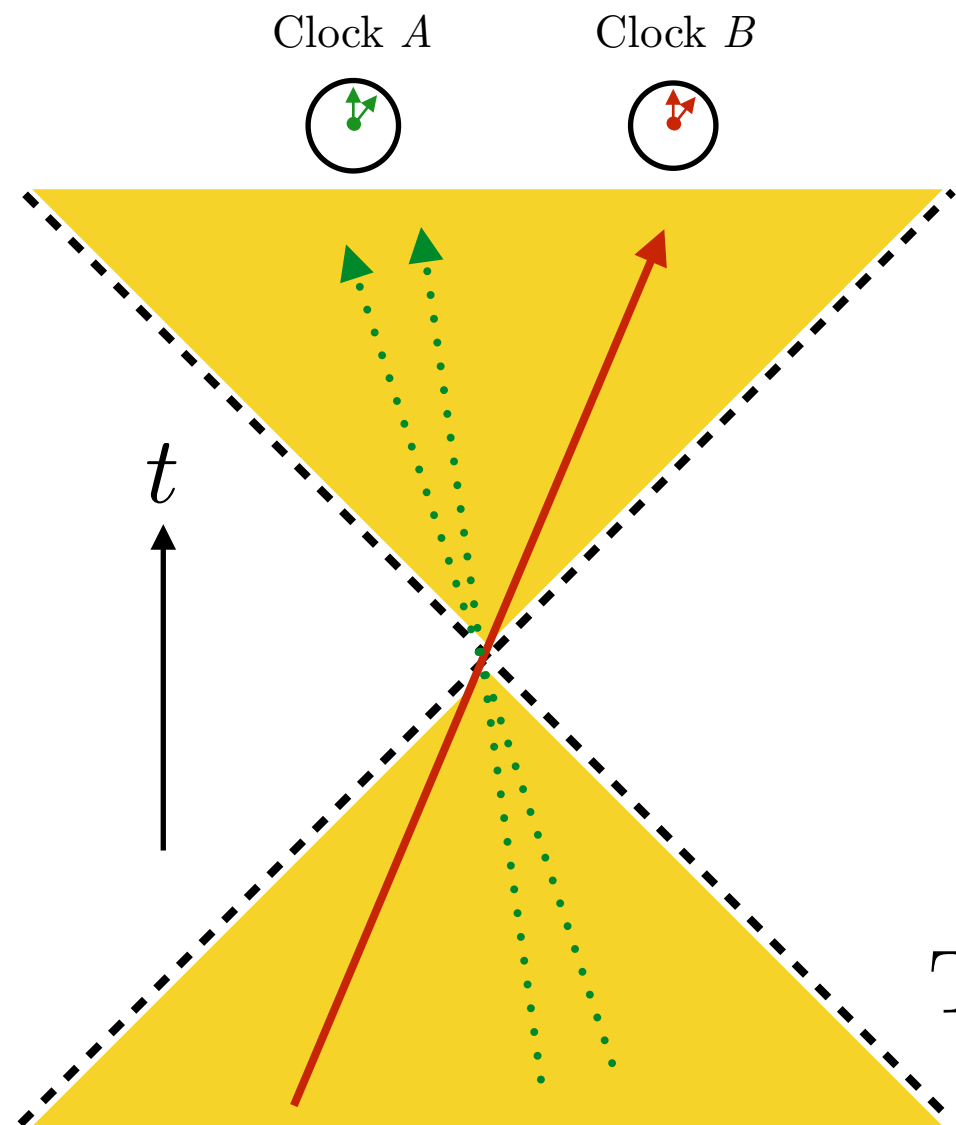
$$= \frac{\langle\langle \psi | E_A(\tau_A) E_B(\tau_B) | \psi \rangle\rangle}{\langle\langle \psi | E_B(\tau_B) | \psi \rangle\rangle}$$

To leading order in Minkowski space the average proper time read by clock A is

$$\langle T_A \rangle = \left(1 - \frac{\langle H_A^{\text{ext}} \rangle - \langle H_B^{\text{ext}} \rangle}{mc^2} \right) \tau_B + \mathcal{O}\left(\left(\frac{1}{mc^2}\right)^2\right)$$

$\frac{\mathbf{P}^2}{2m}$

Classical and quantum time dilation effects



Consider a momentum superposition

$$|\psi_A^{\text{ext}}\rangle = \frac{1}{\sqrt{N}} \left(\cos \theta |\bar{\mathbf{p}}_A\rangle + \sin \theta e^{i\phi} |\bar{\mathbf{p}}'_A\rangle \right)$$

Then the average time read by clock A is

$$\langle T_A \rangle = \left(1 - \underbrace{\frac{\langle H_A^{\text{ext}} \rangle - \langle H_B^{\text{ext}} \rangle}{mc^2}}_{K_{\text{classical}} + K_{\text{quantum}}} \right) \tau_B + \dots$$

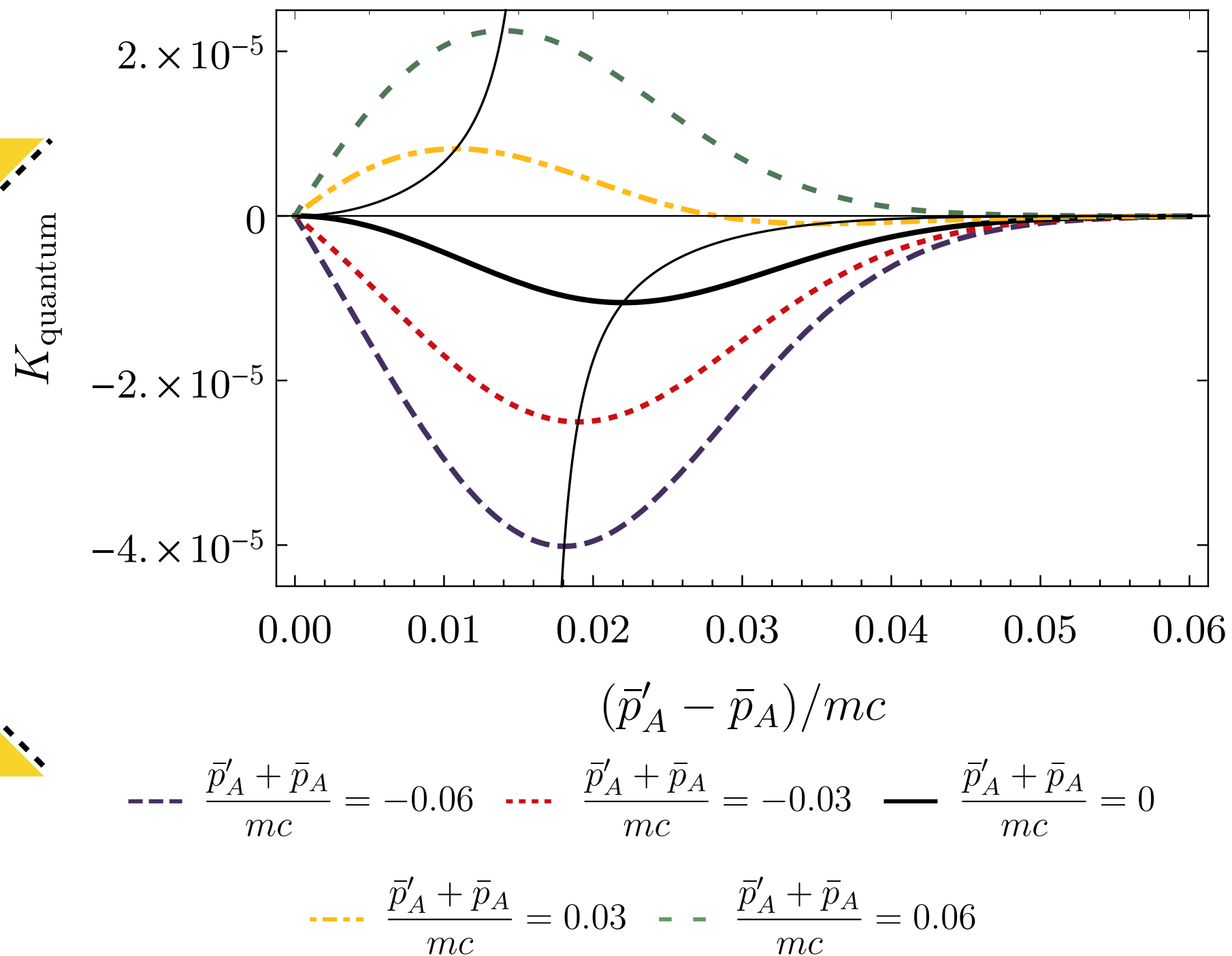
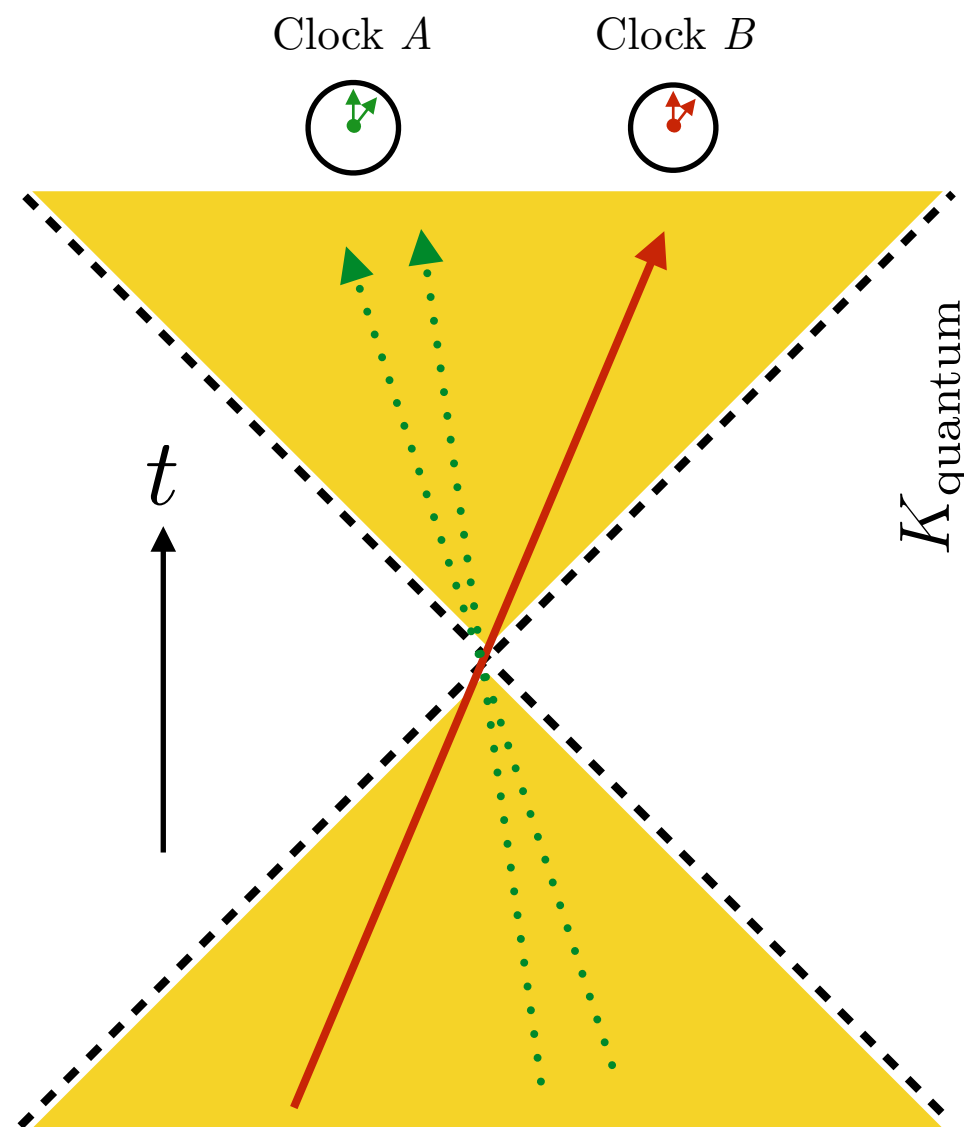
$K_{\text{classical}} + K_{\text{quantum}}$

The classical and quantum contributions are

$$K_{\text{classical}} = \frac{\bar{\mathbf{p}}_A^2 \cos^2 \theta + \bar{\mathbf{p}}'_A{}^2 \sin^2 \theta - \bar{\mathbf{p}}_B^2}{2m^2 c^2} + \frac{\Delta_A^2 - \Delta_B^2}{4m^2 c^2}$$

$$K_{\text{quantum}} = \frac{\sin 2\theta \cos \phi}{8m^2 c^2 N} e^{-\frac{(\bar{\mathbf{p}}'_A - \bar{\mathbf{p}}_A)^2}{4\Delta_A^2}} \left[2 (\bar{\mathbf{p}}'_A{}^2 - \bar{\mathbf{p}}_A^2) \cos 2\theta - (\bar{\mathbf{p}}'_A - \bar{\mathbf{p}}_A)^2 \right]$$

Quantum time dilation effects



$$K_{\text{quantum}} = \frac{\sin 2\theta \cos \phi}{8m^2 c^2 N} e^{-\frac{(\bar{p}'_A - \bar{p}_A)^2}{4\Delta_A^2}} \left[2 (\bar{p}'_A{}^2 - \bar{p}_A^2) \cos 2\theta - (\bar{p}'_A - \bar{p}_A)^2 \right]$$

Is quantum time dilation observable?

1. Time dilation observed for clocks moving several meters per second

C. W. Chou, D. B. Hume, T. Rosenband, D. J. Wineland,
Optical Clocks and Relativity, Science 329, 1630 (2010)

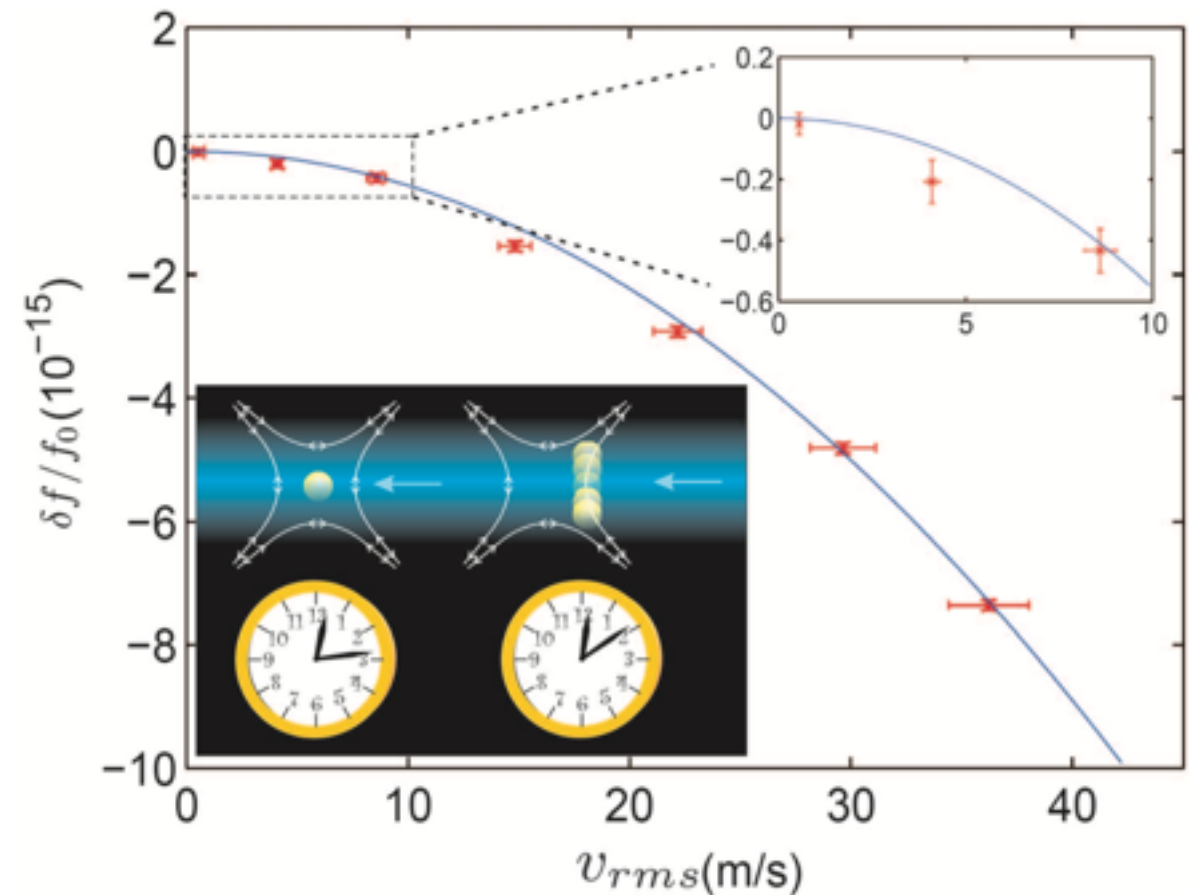
2. Superpositions of atoms moving at speeds that differ by several meters per second has been realized

P. R. Berman, *Atom Interferometry* (Academic Press, 1997)
P. Cladé, S. Guellati-Khélifa, F. Nez, and F. Biraben, Phys. Rev. Lett. 102, 240402 (2009)

3. An estimate of the quantum contribution is $K_{\text{quantum}} \approx 10^{-15}$

4. Given the resolution of atomic clocks, the coherence time of the superposition must be ~ 10 seconds to observe quantum time dilation

T. Kovachy, P. Asenbaum, C. Overstreet, C. A. Donnelly, S. M. Dickerson, A. Sugarbaker, J. M. Hogan and M. A. Kasevich, *Quantum superposition at the half-metre scale*, Nature 528, 530–533 (2015)

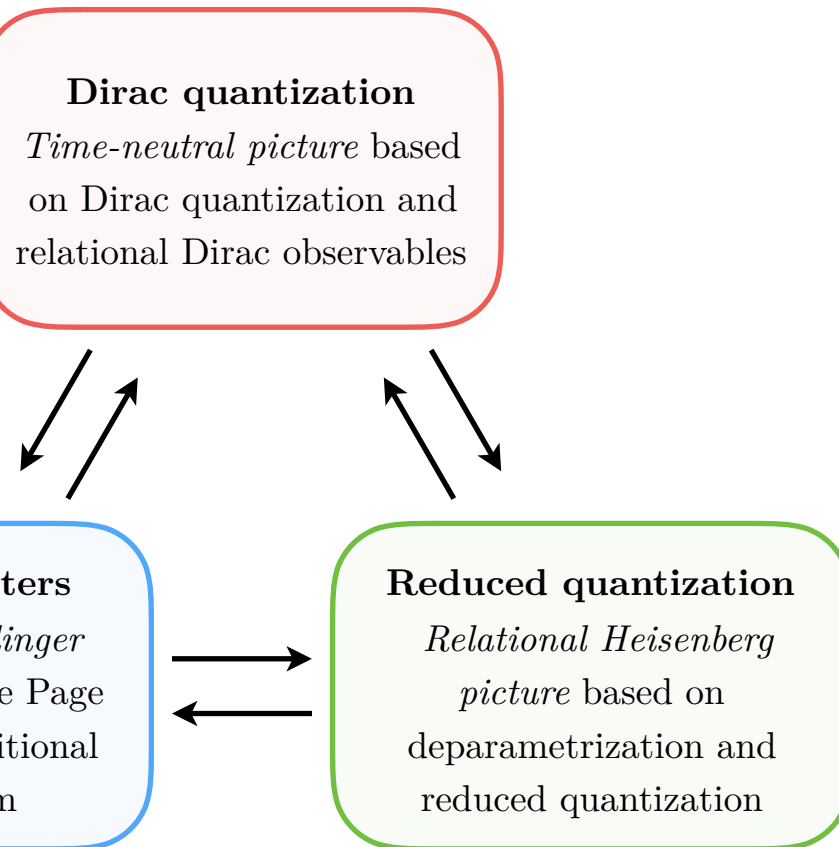


It appears that quantum time dilation may be observable with present day technology!

The Trinity of Relational Quantum Dynamics

A new attack on the problem of time:

The challenges of one approach can be met
with the strengths and tools of another!



1. Normalization of the physical state

$$\langle\langle\psi|\psi\rangle\rangle_{\text{PW}} := \langle\langle\psi| (|t\rangle\langle t| \otimes I_S) |\psi\rangle\rangle$$

is equivalent to

$$\langle\langle\psi|\psi\rangle\rangle_{\text{Phys}} := \langle\langle\psi| \left(\int ds e^{-isH} \right) |\psi\rangle\rangle$$

2. Changing quantum reference frames

1. P. A. Hoehn and A. Vanrietvelde, arXiv:1810.04153
2. A. Vanrietvelde, P. A. Hoehn, F. Giacomini, and E. Castro-Ruiz arXiv:1809.00556
3. F. Giacomini, E. Castro-Ruiz, and Č. Brukner, Nat. Commun. 10, 494 (2019)
4. P. A. Hoehn, Universe 5, 116 (2019)
5. E. Castro-Ruiz, F. Giacomini, A. Belenchia, Č Brukner, arXiv:1908.10165

3. Constructing relational Dirac observables

Why reformulate quantum theory this way?

$$|\psi\rangle\rangle = \left(\int dt |t\rangle\langle t| \right) \otimes I_S |\psi\rangle\rangle = \int dt |t\rangle |\psi_S(t)\rangle$$

Quantum gravity demands that we do!

Ockham's Razor:

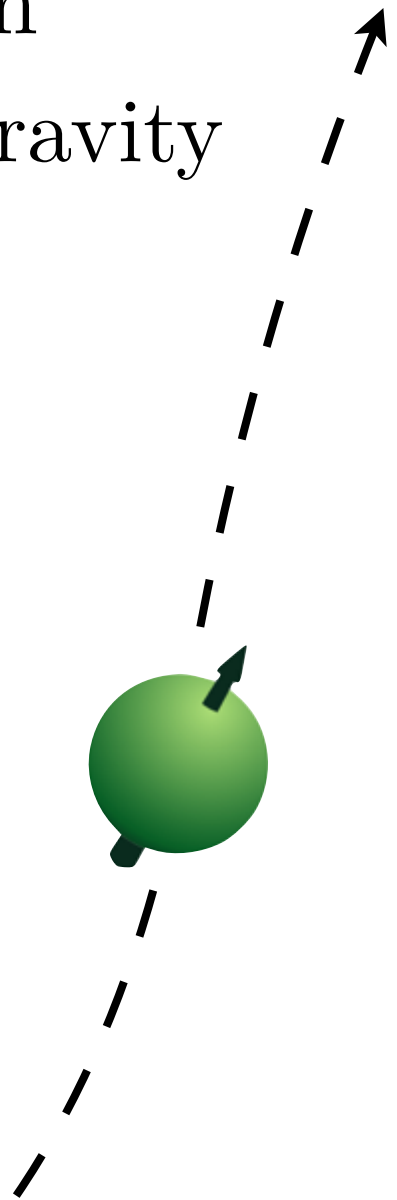
'One motivation for considering such a "condensation" of history is the desire for economy as regards the number of basic elements of the theory: quantum correlations are an integral part of quantum theory already; so one is not adding a new element to the theory. And yet an old element, time, is being eliminated, becoming a secondary and even approximate concept.'

W. K. Wootters, *"Time" Replaced by Quantum Correlations* (1984)

V. Giovannetti, S. Lloyd, and L. Maccone, Quantum Time, *Phys. Rev. D* 92, 045022 (2015)

Summary and outlook

- Generalization to interacting clock and system
 - This results in a time-nonlocal Schrödinger equation
 - Eg. Time dependent Hamiltonian and Newtonian gravity
- Relativistic quantum clocks observe classical and quantum time dilation
- What's next?
 - Experimental proposal for quantum time dilation
 - The trinity of relational quantum dynamics
 - Quantum field theory and quantum gravity
 - Non-ideal clocks and fundamental decoherence



A. R. H. Smith and M. Ahmadi, *Quantizing time: Interacting clocks and systems*, Quantum 3, 160 (2019)

A. R. H. Smith and M. Ahmadi, *Relativistic quantum clocks observe classical and quantum time dilation*, arXiv:1904.12390 [quant-ph] (2019)

P. A. Höhn, M. P. E. Lock, and A. R. H. Smith, *The trinity of relational quantum dynamics*, Forthcoming (2019)