Locally Covariant Quantum Field Theory on Causally Compatible Sets

James Vickers
Joint work with Günther Hörmann,
Yafet Sanchez Sanchez and Christian Spreitzer

University of Southampton

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- Quantisation: The role of Green operators and the causal propagator
- Quantisation: The symplectic form ω (and the space it is defined on)
- Quantisation: CCRs on the space of quasi-local C^*-algebras
- Locally covariant quantum fields and the Haag-Kastler axioms

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- Generalised hyperbolicity for quantum fields
- Relation to other work

Quantum Fields on curved spacetimes

We want to use quantum fields as probes of the singularities of classical spacetimes.

Although not a fundamental theory of nature this provides many insights into important physics. Examples are:

- Hawking radiation by Black Holes
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- How to define the evolution of the fields:
  What is the "causal propagator"?
- How to define the states:
  What are the physical states - what is the vacuum state?

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- Existence and uniqueness of advanced (and retarded) Green operators $G^+ : \mathcal{D}(M) \to C^\infty(M)$ such that
  1. $P \circ G^+ = \mathrm{id}_{\mathcal{D}(M)}$
  2. $G^+ \circ P|_{\mathcal{D}(M)} = \mathrm{id}_{\mathcal{D}(M)}$
  3. $\mathrm{s}upp(G^+\phi) \subset J^+(\mathrm{s}upp(\phi))$ for all $\phi \in \mathcal{D}(M)$

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- Define *physical fields* using micro-local spectrum condition on 2-point function.

Issues in low regularity

- Existence of (weak) solution in $\mathbb{R}^{n+1}$. What regularity is reasonable for g_{ij}? What regularity do we want for our solution?
- Well-posedness of corresponding solution on (M, g). Global hyperbolicity in non-smooth case. Causality results in non-smooth case. Higher order energy-estimates in non-smooth case.
- Choice of function spaces for Green operators. Now a map between Sobolev spaces.
- Symplectic form ω - What space is this defined on? Exact sequence result - Does this hold in non-smooth case?
- Haag-Kastler axioms - need causality results in low regularity setting
- Micro-local condition for physical states. Needs Sobolev micro-local analysis. What Sobolev space works mathematically/physically?

Our choice of regularity

We take the metric g to be $C^{1,1}(M)$

- Curvature in L^∞ but allows for jumps in energy-momentum tensor at an interface.
- Minimal condition that ensures existence and uniqueness of solutions to the geodesic equation.
- Causality results (including global hyperbolicity) go through more or less unchanged from smooth case [KSSV]
- Solutions of the wave equation are in $H^2_{loc}(M)$

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- Use compactness argument to show that the generalised solution converges to a (weak) solution $u^+ \in H^2_{loc}(\mathbb{R}^{n+1})$ of the original equation.
- Use $J^+_\epsilon(U) \subset J^+(U)$ to show that for zero initial data $\mathrm{s}upp(u^+) \subset J^+(\mathrm{s}upp(f))$

Solutions in (M, g)

Causal Structure:

For $C^{1,1}$ spacetimes the smooth definition(s) of global hyperbolicity remain unchanged. It is also shown by Minguzzi that for such spacetimes one may choose a *smooth* temporal function.

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H^1 Energy inequality:

Use divergence theorem on energy-momentum tensor on $U^+ := U \cap J^+(\Sigma)$ to show

$$||u||_{\tilde{H}^1(\Sigma_\tau \cap U^+)} \leq K \left(||u||_{\tilde{H}^1(\Sigma_0 \cap U^+)} + ||f||_{L^2(U_\tau)} \right)$$

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Uniqueness:

Use energy inequality to show that $H^2_{loc}(M)$ solutions of $\Box_g u = f$ with $f \in H^1_{loc}(M)$ and initial data in $H^2(\Sigma) \times H^1(\Sigma)$ are unique.

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Causal support:

Use energy inequality on $U = J^-(p) \cap J^+(\Sigma)$ for $p \in M \setminus J^+(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f))$ to show that $\mathrm{supp}(u) \subset J(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f))$.

Solutions in (M, g) (cont.)

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Existence for compact source and initial data:

Follow proof in Ringström "The Cauchy problem in General Relativity" to piece together local solutions on $\mathbb{R}^{n+1}$ to obtain a (weak) solution $u \in C^0(\mathbb{R}, H^2(\Sigma_t)) \cap C^1(\mathbb{R}, H^1(\Sigma_t)) \cap H^2_{loc}(M)$ to the initial value problem. Note use of temporal function to avoid need to take traces in Sobolev spaces.

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Global Existence and Uniqueness:
Let p be any point to the future of Σ. Then $K_p = J^-(p) \cap J^+(\Sigma)$ is a compact set. We now use local existence on all such sets together with the global uniqueness result to give a global solution. We also use Causal support result to show that $\mathrm{s}upp(u) \subset J(\mathrm{s}upp(u_0) \cup \mathrm{s}upp(u_1) \cup \mathrm{s}upp(f))$.

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Higher order Energy estimates:
We obtain these from local estimates using a partition of unity

$$||u||_{C^0([0,T],H^2(\Sigma))}+||u||_{C^1([0,T],H^1(\Sigma))} \leq C\Big(\|u_0\|_{H^2(\Sigma)}+\|u_1\|_{H^1(\Sigma)}+||f||_{H^1(M)}\Big)$$

Existence and Uniqueness

Theorem: Global Existence and Uniqueness

Let (M, g) be a connected, oriented, time oriented $(n+1)$-dimensional Lorentzian globally hyperbolic manifold with $C^{1,1}$ metric and Σ a smooth spacelike n-dimensional Cauchy hypersurface. Let t be a (smooth) temporal function with $t^{-1}(0) = S$ and let n be the future directed timelike unit normal to Σ.
Given initial data $(u_0, u_1) \in H^2(\Sigma) \times H^1(\Sigma)$ and source $f \in L^2_{loc}(\mathbb{R}, H^1(\Sigma_t))$ then there exists a unique (weak) solution $u \in C^0(\mathbb{R}, H^2(\Sigma_t)) \cap C^1(\mathbb{R}, H^1(\Sigma_t) \cap H^2_{loc}(M)$ to the initial value problem

$$\begin{aligned} \Box_g u &= f \quad \text{on } M, \\ u &= u_0 \quad \text{on } \Sigma, \\ \nabla_n u &= u_1 \quad \text{on } \Sigma. \end{aligned}$$

Moreover $\mathrm{supp}(u) \subset J(\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \cup \mathrm{supp}(f))$.

Well-posedness

Using the higher-order energy estimate we can establish the following spacetime energy estimate

$$||u||_{H^2(K)} \leq C\Big(\|u_0\|_{H^2(\Sigma)} + \|u_1\|_{H^1(\Sigma)} + ||f||_{H^1(M)}\Big).$$

for any compact $K \subset M$

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Corollary: Well-posedness

The solution to the Cauchy problem described in the above Theorem is well-posed in the sense that the *solution map*

$$\begin{array}{rcl} Sol : H^2(\Sigma) \times H^1(\Sigma) \times H^1_{comp} & \to & H^2_{loc} \\ (\ u_0, \quad u_1, \quad f\ ) & \mapsto & u \end{array}$$

is continuous.

Function spaces for our Green Operators

The definition of the Green operators in the non-smooth setting will require us to choose suitable function spaces as domain and range.

We therefore define the following spaces

$$\begin{aligned} V_0 =& \{\phi \in H^2_{comp}(M) \text{ such that: } \Box_g \phi \in H^1_{comp}(M)\} \\ U_0 =& H^1_{comp}(M) \\ V_{sc} =& \{\phi \in H^2_{loc}(M) \text{ such that: } \Box_g \phi \in H^1_{loc}(M) \\ & \text{and } \operatorname{supp}(\phi) \subset J(K) \text{ where } K \Subset M\} \end{aligned}$$

V_0 are the compactly supported functions contained in the space V given by the graph norm of $\Box_g$,

$$\|\psi\|_{Gr} := \|\psi\|_{H^2(M)} + \|\Box_g \psi\|_{H^1(M)}$$

while V_{sc} are functions in V satisfying the causal support condition.

Green Operators

Definition:

A linear map

$$G^+ : H^1_{comp}(M) \to H^2_{loc}(M)$$

satisfying the following properties.

1. $\Box_g G^+ = id_{H^1_{comp}(M)}$
2. $G^+ \Box_g|_{V_0} = id_{V_0}$
3. $\mathrm{s}upp(G^+(f)) \subset J^+(\mathrm{s}upp(f))$ for all $f \in H^1_{comp}(M)$

is an advanced Green operator for $\Box_g$.

A retarded Green operator G^- is defined similarly.

Note: Our function spaces $H^2_{\mathrm{loc}}(M)$ and $H^1_{\mathrm{comp}}(M)$ used as target space and domain for the Green operators **do not depend on a background metric** and are in perfect accordance with the theory of so-called *regular fundamental solutions* for hyperbolic operators with constant coefficients (Hörmander).

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Exact sequence

The causal propagator $G = G^+ - G^-$ is a map $G : H^1_{comp}(M) \to H^2_{loc}(M)$ and the following complex is exact.

$$0 \longrightarrow V_0 \xrightarrow{P} U_0 \xrightarrow{G} V_{sc} \xrightarrow{P} H^1(M)$$

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Skew symmetry

Given $\chi, \varphi \in H^1_{comp}(\Sigma)$ we have that

$$\int_M (G^+(\chi)\varphi)\nu_g = \int_M (\chi G^-(\varphi))\nu_g$$

Causally Compatible subsets

Definition

A subset $\Omega \subset M$ in a time-oriented Lorentzian manifold is called *causally compatible* if for all points $x \in \Omega$ we have

$$J^{\pm}_{\Omega}(x) = J^{\pm}_{M}(x) \cap \Omega \quad \forall x \in \Omega$$

In other words whenever two points in Ω can be joined by a causal curve in M then this can also be done within Ω.

It followsfrom the defintion that for each subset $A \in \Omega$ we have

$$J^{\pm}_{\Omega}(A) = J^{\pm}_{M}(A) \cap \Omega \quad \forall x \in \Omega$$

Note: Being causally compatible is transitive. So that if $\Omega \subset \Omega' \subset \Omega''$ with Ω a causally compatible subset of Ω' and Ω' a causally compaptible subset of Ω'' then Ω is a causally compatible subset of Ω''.

Restrictions of G to causally compatible subsets

Theorem

Let M be a time oriented connected globally hyperbolic manifold with a $C^{1,1}$ Lorentzian metric. Let G^{+}be the Green operators for $\Box_g$. Let $\Omega \subset M$ be a causally compatible open subset.

Then for all $\varphi \in H^1_{comp}(M)$ with $\mathrm{s}upp(\varphi) \subset \Omega$

$$\tilde{G}^{+}(\varphi) := G^{+}(\varphi_{ext})|_{\Omega}, \quad \text{where } \varphi_{ext} \text{ denotes extension by zero.}$$

is an advanced Green operator for the restriction of $\Box_g$ to Ω. We denote the restriction of $\Box_g$ to Ω by $\widetilde{\Box}_g$.

Notice that $\forall u \in H^2_{loc}(M)$ we have $\widetilde{\Box}_g(u|_{\Omega}) = \Box_g|_{\Omega}(u|_{\Omega}) = (\Box_g u)|_{\Omega}$ and $\forall u \in H^2(\Omega)$ with $\mathrm{s}upp(u) \subseteq \Omega$ we have $(\widetilde{\Box}_g u)_{ext} = \Box_g(u_{ext})$.

The symplectic space

Define $\tilde{\omega} : U_0 \times U_0 \to \mathbb{R}$ by

$$\tilde{\omega}(\phi, \psi) = \int_M G(\phi)\psi\nu_g$$

where $G = G^+ - G^-$ is the causal propagator. Then ω is bi-linear and skew-symmetric. However, $\tilde{\omega}$ is degenerate because $ker(G)$ is different from zero. Moreover from the *exact sequence* result we have that

$$ker(G) = \Box_g V_0.$$

Therefore on the quotient space $V(M) = U_0/ker(G) = U_0/\Box_g V_0$ the degenerate form $\tilde{\omega}$ induces a symplectic form which we denote by ω.

Symplectic Space

$V(M) = U_0/\Box_g(V_0)$ equipped with the bi-linear form ω induced on the quotient space by $\tilde{\omega}$ is a *symplectic vector space*

Compatibitly of Green Operators

The previous result together with the uniqueness of Green operators shows that Green operators are compatible on causally compatible subsets.

Theorem

Let $\iota : M_1 \to M_2$ be a time-orientation preserving isometric embedding (so that $\iota(M_1) \subset M_2$ is a causally compatible open subset) and let $H^1_{comp}(M_i)$ be the set of $\phi \in H^1_{comp}(M)$ with $\mathrm{s}upp(\phi) \subset M_i$.
Then the following diagram commutes

$$\begin{array}{ccc} H^1_{comp}(M_1) & \xrightarrow{ext} & H^1_{comp}(M_2) \\ G_1^{\pm}\downarrow & & \downarrow G_2^{\pm} \\ H^2_{loc}(M_1) & \xleftarrow{res} & H^2_{loc}(M_2) \end{array}$$

Remark: This shows that ι also induces a symplectic linerar map

$$S : (V(M_1), \omega(G_1)) \to (V(M_2), \omega(G_2))$$

Weyl systems and the CCR representation of $(V(M), \omega)$

The remaining results are virtually identical to the smooth case (see BGP)

We now introduce the definition of a Weyl system and a CCR- representation of (V, ω) for a general symplectic space.

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Weyl System

A *Weyl system* of the symplectic vector space (V, ω) consists of a C^*-algebra $\mathcal{A}$ with unit and a map $W : V \to \mathcal{A}$ such that for all φ, ψ

1. $W(0) = 1$,
2. $W(-\varphi) = W(\varphi)^*$
3. $W(\varphi) \cdot W(\psi) = e^{-i\omega(\varphi,\psi)/2} W(\varphi + \psi)$

Constructing a Weyl system on (V, ω)

Let $H = L^2(V, \mathbb{C})$ be the Hilbert space of square-integrable complex-valued functions on V with respect to the counting measure, i.e., H consists of those functions $F : V \to \mathbb{C}$ that vanish everywhere except for countably many points and satisfy $||F||^2_{L^2} := \sum_{\phi \in V} |F(\phi)|^2 < \infty$

The Hermitian product on H is given by

$$(F, G)_{L^2} = \sum_{\phi \in V} \overline{F(\phi)} \cdot G(\phi) \tag{1}$$

Let $\mathcal{A} := \mathcal{L}(H)$ be the C^* algebra of bounded linear operators on H .
We define the map $W : V \to \mathcal{A}$ by

$$(W(\phi)F)(\psi) := e^{\frac{i\omega(\phi,\psi)}{2}} F(\phi + \psi) \tag{2}$$

Then $W : V \to \mathcal{A}$ is a Weyl system for (V, ω)

Canonical Commutation Relations

CCR-representation

A Weyl system $(\mathcal{A}, W)$ of a symplectic vector space (V, ω) is called a *CCR-representation* of (V, ω) if $\mathcal{A}$ is generated as a C^*- algebra by the elements $W(\varphi)$, $\varphi \in V$. In this case we call $\mathcal{A}$ a CCR-algebra of (V, ω).

Remark: Of course , for any Weyl system $(\mathcal{A}, W)$ we can simply replace $\mathcal{A}$ by the C^*-subalgebra generated by the elements $W(\varphi), \varphi \in V$ and we obtain a CCR-representation.

We hve therefore shown how to generate a Weyl system and a CCR-representation for an arbitrary symplectic vector space (V, ω) and hence in particular for $(V(M), \omega)$ constructed via the Green operator. Uniqueness also holds (in an appropriate sense).

Compatibility of CCR-algebras

Let (V_1, ω_1) and (V_2, ω_2) be two symplectic vector spaces and let $S : V_1 \to V_2$ be a symplectic linear map. Then there exist a unique injective $*$-morphism

$$CCR(S) : CCR(V_1, \omega_1) \to CCR(V_2, \omega_2) \tag{3}$$

such that the diagram below commutes

$$\begin{array}{ccc} V_1 & \xrightarrow{\quad S \quad} & V_2 \\ {\scriptstyle W_1}\downarrow & & \downarrow{\scriptstyle W_2} \\ CCR(V_1, \omega_1) & \xrightarrow{CCR(S)} & CCR(V_2, \omega_2) \end{array}$$

Remark: In particular if $\iota : M_1 \to M_2$ is a time-orientation preserving isometric embedding we may apply this result to

$$S : (V(M_1), \omega(G_1)) \to (V(M_2), \omega(G_2))$$

to show the compatibility of the CCR-algebras for causally compatible sets.

Quasi-local C^*-algebras

Definition:

A bosonic quasi-local C^*-algebra is a pair $(\mathcal{U}, \{\mathcal{U}_\alpha\}_{\alpha\in I})$ of a C^*- algebra $\mathcal{U}$ and a family $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of C^*-subalgebras, where I is a directed set with orthogonality relation such that the following holds:

- $\mathcal{U}_\alpha \subset \mathcal{U}_\beta$ whenever $\alpha \leq \beta$
- $\mathcal{U} = \overline{\cup\mathcal{U}_\alpha}$ where the bar denotes closure in the $\mathcal{U}$ norm.
- The algebras $\mathcal{U}_\alpha$ have a common unit $\mathbb{1}$.
- If $\alpha \perp \beta$ the commutator of $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ is trivial: $[\mathcal{U}_\alpha, \mathcal{U}_\beta] = \{0\}$

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In our situation we take:

$$I =: \{\mathcal{O} \subset M : \mathcal{O} \text{ is open, relatively compact, causally compatible,} \\ \text{and globally hyperbolic}\} \cup \{\emptyset, M\}$$

On I we take the inclusion $\subseteq$ as the partial order $\leq$ and define the orthogonality relation by $\mathcal{O} \perp \mathcal{O}' :\iff J(\overline{\mathcal{O}}) \cap \overline{\mathcal{O}'} = \emptyset$.
So that they are orthogonal iff they are *causally independent*.

Constructing the quasi-local C^*-algebra

We define the index set I as above.

For any non-empty $\mathcal{O} \in I$ we restrict $\Box_g$ to $\mathcal{O}$.

Due to the causal compatibility of $\mathcal{O} \in M$ the restrictions of the Green operators G^+ and G^- to $\mathcal{O}$ yield Green operators $G^+_{\mathcal{O}}$ and $G^-_{\mathcal{O}}$ for $\Box_g|_{\mathcal{O}}$

Let $(V(\mathcal{O}), \omega_{\mathcal{O}})$ be the corresponding symplectic vector space where $\omega_{\mathcal{O}} = \omega|_{\mathcal{O}}$.

Let $\mathcal{U}_{\mathcal{O}}$ be the CCR-representation of the symplectic space $(V(\mathcal{O}), \omega_{\mathcal{O}})$.

This is a C^*-subalgebra of $\mathcal{U}_M$ and $\mathcal{U}_{\mathcal{O}}$ is a C^*-subalgebra of $\mathcal{U}_{\mathcal{O}'}$ if $\mathcal{O} \subset \mathcal{O}'$.

Proposition

$$\mathcal{U}_M = C^* \left(\bigcup_{\mathcal{O} \in I} \mathcal{U}_{\mathcal{O}} \right)$$

and $(\mathcal{U}_M, \{\mathcal{U}_{\mathcal{O}}\}_{\mathcal{O} \in I})$ is a weak quasi-local C^*-algebra.

The Haag-Kastler Axioms

We have shown how to go from a globally hyperbolic $C^{1,1}$ spacetime (M, g) to the space of quasi-local C^*-algebras $(\mathcal{U}_M, \{\mathcal{U}_\mathcal{O}\}_{\mathcal{O}\in I})$ generated by the symplectic vector spaces $(V(\mathcal{O}), \omega_\mathcal{O})$ coming from the causal propagator $G_\mathcal{O}$ for $\Box_g|_\mathcal{O}$.

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This construction satisfies the Haag-Kastler axioms:

1. If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{U}_{\mathcal{O}_1} \subset \mathcal{U}_{\mathcal{O}_2}$ for all $\mathcal{O}_1, \mathcal{O}_2 \in I$
2. $\mathcal{U}_M = \overline{\cup \mathcal{U}_\mathcal{O}}$
3. $\mathcal{U}_M$ is simple
4. The $\mathcal{U}_\mathcal{O}$'s have a common unit $\mathbb{1}$
5. For all $\mathcal{O}_1, \mathcal{O}_2 \in I$ with $J(\overline{\mathcal{O}_1}) \cap \overline{\mathcal{O}_2} = \emptyset$, the $\mathcal{U}_{\mathcal{O}_1}, \mathcal{U}_{\mathcal{O}_2}$ commute.
6. (Time-slice axiom) Let $\mathcal{O}_1 \subset \mathcal{O}_2$ be nonempty element of I admitting a common Cauchy hypersurface Σ. Then $\mathcal{U}_{\mathcal{O}_1} = \mathcal{U}_{\mathcal{O}_2}$
7. Let $\mathcal{O}_1, \mathcal{O}_2 \in I$ and let the Cauchy development $D(\mathcal{O}_2)$ be relatively compact in M. If $\mathcal{O}_1 \subset D(\mathcal{O}_2)$, then $\mathcal{U}_{\mathcal{O}_1} \subset \mathcal{U}_{\mathcal{O}_2}$.

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The only problematic issue is the time-slice axiom but the $C^{1,1}$-causality results plus properties of temporal function ensure that the proof in the

Topological Issues

Recall that G is a linear map $U_0 \to V_{sc}$ and let G_0 denote the associated map from the quotient $U_0/\ker(G)$ to $\mathrm{im}(G) \subseteq V_{sc}$, defined by $G_0(\phi + \ker(G)) := G\phi$ for every $\phi \in U_0$, which is linear and bijective by construction. We therefore have the following chain of **algebraic** isomorphisms of vector spaces

$$U_0/\Box_g V_0 = U_0/\ker(G) \cong \mathrm{im}(G) = \ker(\Box_g) \subseteq V_{sc}.$$

The qustion arises if $U_0/\ker(G) \cong \mathrm{im}(G)$ is a topological isomorphism.

Proposition

Let the quotient $U_0/\ker(G)$ be equipped with the finest topology such that the canonical surjection $\pi\colon U_0 \to U_0/\ker(G)$, $\phi \mapsto \phi + \ker(G)$ is continuous. Then $U_0/\ker(G) \cong \mathrm{im}(G)$ is a **topological** isomorphism.

Relation to other work

We may define a symplectic form Ξ on $\ker(\Box_g)$ by

$$\Xi(\Psi_1, \Psi_2) =: \int_{\Sigma_t} (\pi_1\varphi_2 - \pi_2\varphi_1)\nu_{g_t}$$

where φ_i and π_i are $\Psi_i|_\Sigma$ and $\nabla_n\Psi_i|_\Sigma$ respectively. This defintion is indenpendent of the choice of smooth spacelike Cauchy surface Σ.

Note: We have a topological isomorphism $\ker(\Box_g) = \mathrm{im}(G) \cong U_0/\ker(G)$.

Proposition

$$\Xi(\Psi_1, \Psi_2) = \omega([\psi], [\phi])$$

where Ψ and $[\phi]$ are related by the above topological isomorphism.

So $(V(M), \omega)$ is equivalent to using $(\ker(\Box_g), \Xi)$ as is done by Wald.
In this description the Weyl relation:
$W(f) \cdot W(g) = e^{-i\omega(f,g)/2} W(f+g)$ translates to $[\hat{\varphi}(x), \hat{\pi}(y)] = \delta(x-y)$

The Physical States

To obtain the standard description in terms of operators in Hilbert space one needs to specify states. Note we cannot use the standard Fock space approach since without the symmetry of Minkowski space we do not have a preferred vacuum.

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In the smooth case one uses Hadamard states defined via the wavefront set of the two-point function to specify the physical states.
In the non-smooth case one needs to use instead a Sobolev Wavefront set to define adiabatic states.

This is the topic for a separate talk!

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