Black holes and superradiant instabilities

Time-domain study of the scalar perturbations in Kerr metric

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Introduction

Goal

Study the stability of black hole (BH) solutions under (scalar) perturbations

Results in General Relativity (GR) for rotating BHs:

- massless: stable[1]
- confined perturbations[2]: superradiant instability

... beyond classical results?

- Scalar with non-trivial self-coupling?
- Non-minimally coupled scalar?
- Beyond-GR BH solutions?

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[1] Press,Teukolsky,ApJ, Vol. 185, pp. 649-674 (1973)

[2] Brito,Cardoso,Pani, arxiv:1501.06570v3

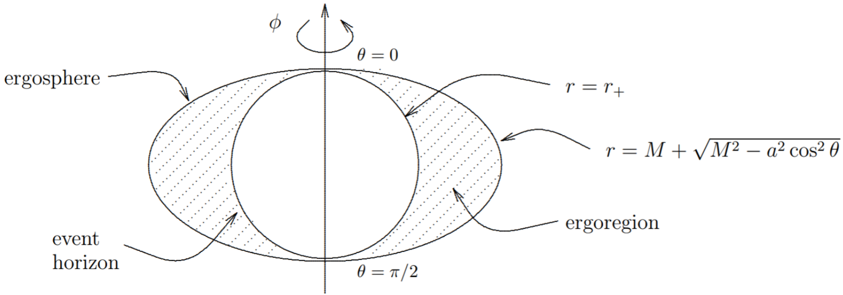
BH superradiance: rotating black holes

- Kerr metric

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})dt^2 - \frac{4aMr}{\rho^2}\sin^2\theta dt d\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma}{\rho^2}\sin^2\theta d\phi^2 \qquad (1)$$

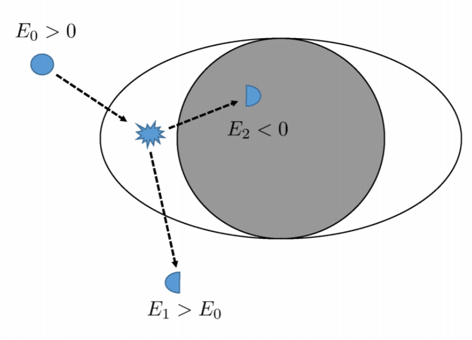
$$\rho^2 \equiv r^2 + a^2\cos^2\theta \qquad \Delta \equiv r^2 + a^2 - 2Mr \qquad \Sigma \equiv (r^2 + a^2)^2 - a^2\Delta\sin^2\theta \qquad (2)$$

- stationary and axis-symmetric: $k_t^\alpha = \frac{\partial x^\alpha}{\partial t}$, $k_\phi^\alpha = \frac{\partial x^\alpha}{\partial \phi}$
- ergoregion: defined by the killing horizon of k_t



BH superradiance: Penrose process

$E \equiv -k_t^\alpha p_\alpha \quad L \equiv k_\phi^\alpha p_\alpha$



$E < 0$ allowed inside ergoregion! Energy extraction, $E_1 > E_0$

$$\delta J \leq \frac{\delta M}{\Omega_H} \qquad \delta M = E,\ \delta J = L$$

BH superradiance: Superradiant scattering

- Assume separable wave equation:

$$\frac{d^2\Phi}{dr_*^2} + V_{eff}(r)\Phi = 0 \tag{3}$$

- $k_H^2 \equiv V_{eff}(r \to r_+) \quad k_\infty^2 \equiv V_{eff}(r \to \infty)$
- $r \to +\infty \quad \Phi \sim \mathcal{I}e^{+ik_\infty r_*} + \mathcal{R}e^{-ik_\infty r_*}$
- $r \to r_+ \quad \Phi \sim \mathcal{T}e^{-ik_H r_*}$

$$|\mathcal{R}|^2 = |\mathcal{I}|^2 - \frac{k_H}{k_\infty}|\mathcal{T}|^2 \tag{4}$$

Superradiance condition, in Kerr: $$0 < \omega < m\Omega_H \tag{5}$$

Quasi-normal modes

- Characteristic free modes of BH+perturbation system
- eigensolutions to Schroedinger-like equation (3)
- physical boundary conditions: ingoing at r_+, outgoing at ∞: $(\mu = 0)$

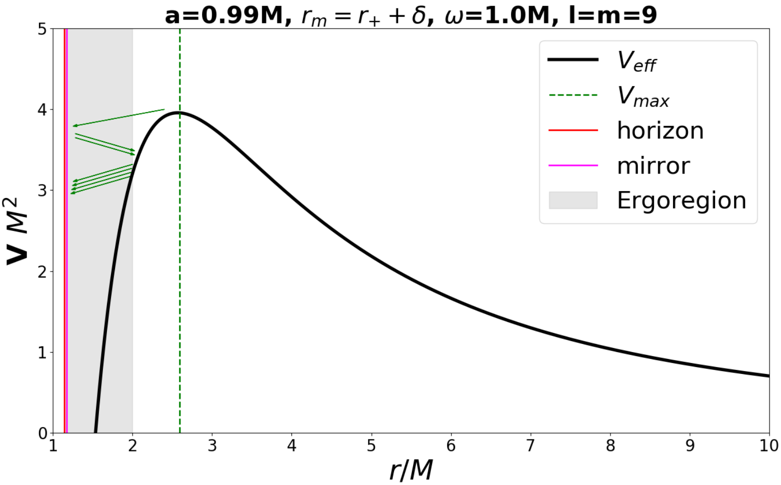
$$\begin{aligned} \Psi &\sim e^{-i\omega t + i(\omega - m\Omega_H) r_*} \qquad r_* \to -\infty \;\; (r \to r_+) \\ &\sim e^{-i\omega(t - r_*)} \qquad r_* \to +\infty \end{aligned}$$

- system is dissipative (horizon!): complex frequencies

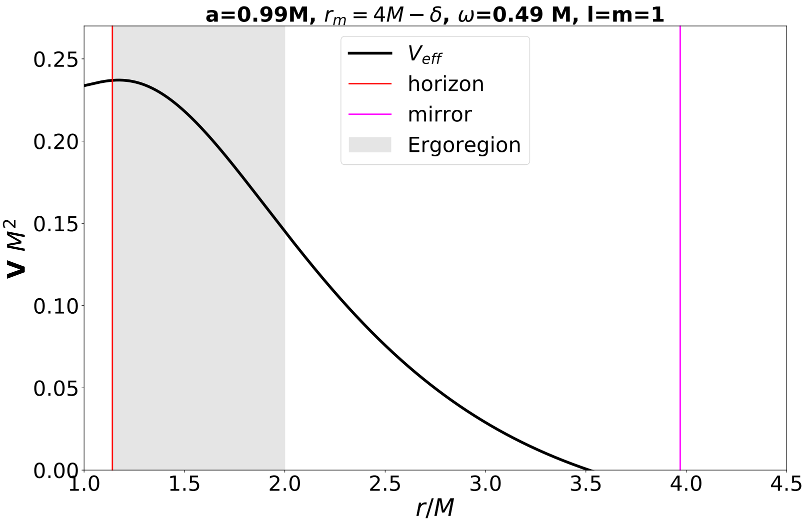
$$\omega = \hat{\omega} + i\nu$$

$\nu < 0$, decay in time
$\nu > 0$, exponential growth $\to$ instability!

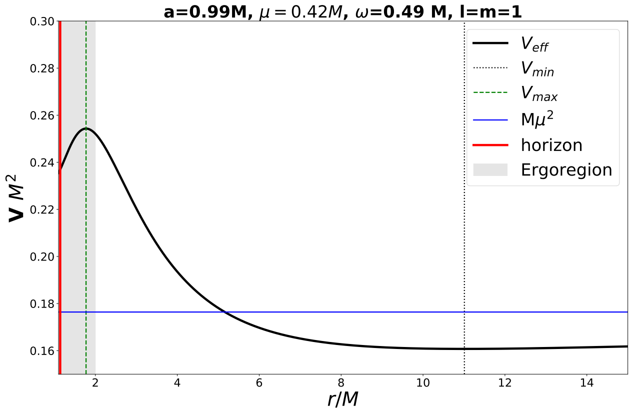
Superradiant instabilities: bh-bomb I



Superradiant instabilities: bh-bomb II



Superradiant instabilities: massive field



- $r_* \to \infty$: $\Psi \sim \mathcal{A} e^{+\sqrt{\mu^2-\omega^2} r_*} + \mathcal{B} e^{-\sqrt{\mu^2-\omega^2} r_*}$
- $\mathcal{A} = 0 \to$ quasi-bound states
- for $\omega \lesssim \mu$, confinement: $\sim \frac{e^{-\mu r}}{r}$

Frequency domain results

Analytical solutions only in specific limits:

- $\mu M \ll 1$: [3]

$$(M\nu_{max}) \simeq \frac{a}{24M}(\mu M)^9 \tag{6}$$

- $\mu M \gg 1$, $(\frac{a}{M} \to 1)$ [4]

$$(M\nu_{max}) \simeq 10^{-7} e^{-1.84\mu M} \tag{7}$$

- exact results for full $\{\mu M, a\}$, resort to numerics
- in ω-domain: WKB approximations, Shooting, Continued fraction...
- when equations are non-separable, t-domain approach

[3] Detweiler, PRD 22, 2323 (1980)

[4] Eardley, Zourous, Annals of Physics 118, 139 (1979)

Numerical t-domain approach[5]

- $1 + 1$D decomposition:

$$\Phi(t, r, \theta, \phi) = \sum_{\ell=|m|}^{\infty} \frac{\psi_{\ell m}(t, r)}{r} e^{im\phi} \mathbf{Y}_{\ell m}(\theta)$$

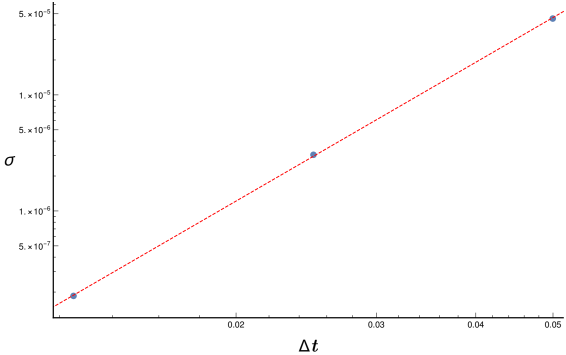
- System of coupled PDEs (ℓ-mode couples to $\ell \pm 2$)

$$S_{\ell m}(\theta) = Y_{\ell m} + \sum_{\ell \neq \ell'} \frac{\langle \ell m | (a^2 \omega^2 \cos \theta^2) | \ell' m' \rangle}{\ell(\ell + 1) - \ell'(\ell' + 1)} Y_{\ell' m} + \cdots$$

- Method of lines with fourth-order Runge-Kutta time step and O(4) finite difference scheme for spatial derivatives
- Boundary conditions
- Ingoing, $r \to r_+$
- ($\mu = 0$) Outgoing, $r \to \infty$
- ($\mu \neq 0$) Bounded solution, $r \to \infty$

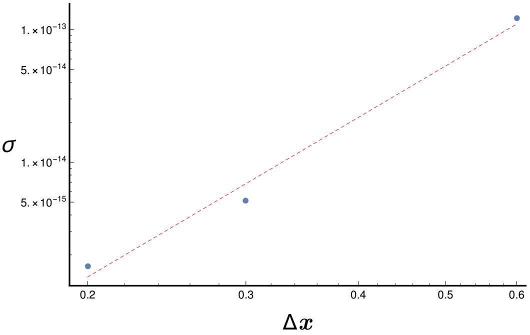
[5] Dolan,arXiv:1212.1477[gr-qc](2013)

Numerical convergence: Δt resolution



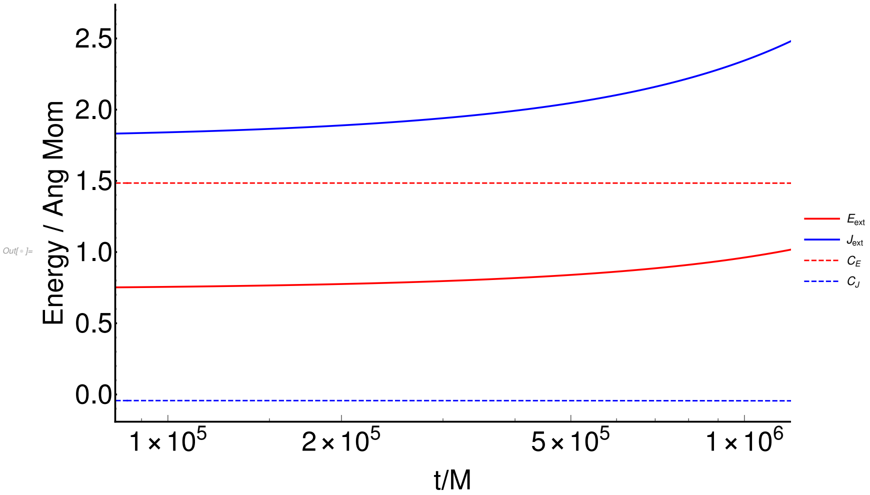
$$\log \sigma = (1.93 \pm 0.15) + (3.97 \pm 0.04) \log \Delta t$$

Numerical convergence: Δx resolution

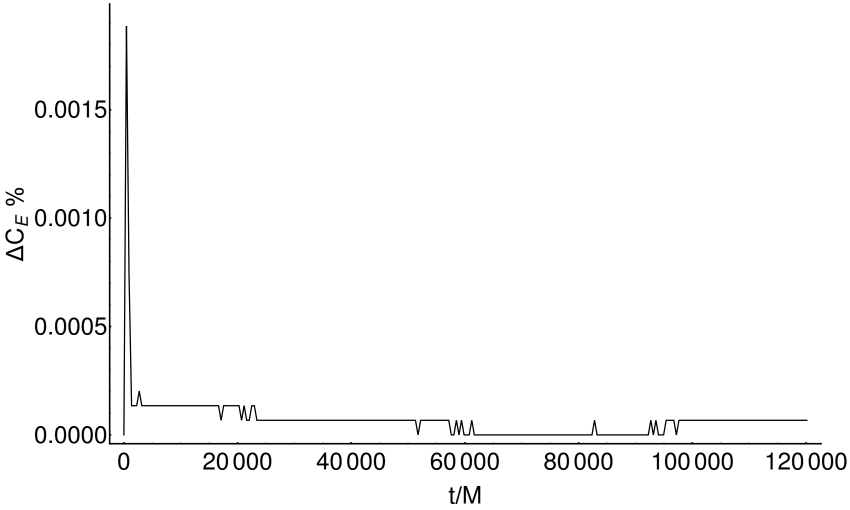


$$\log \sigma = (-27.80 \pm 0.56) + (3.99 \pm 0.47) \log \Delta x$$

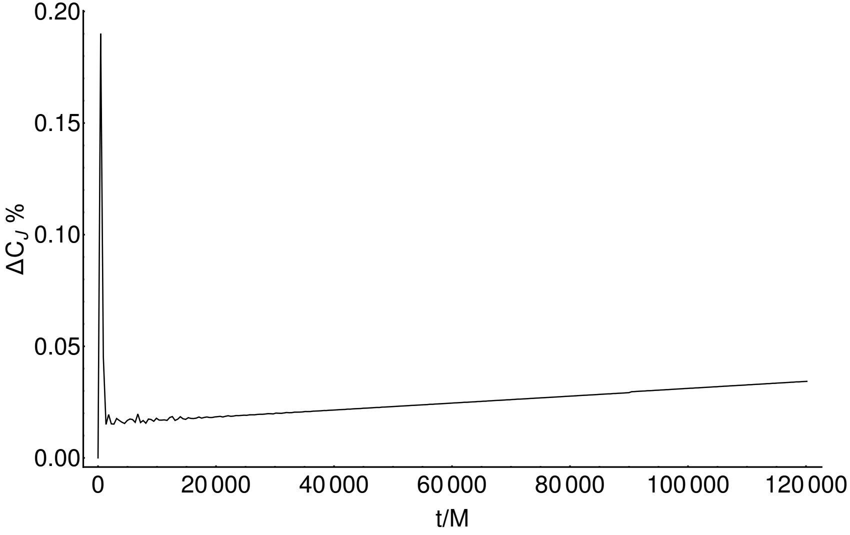
E/J conservation



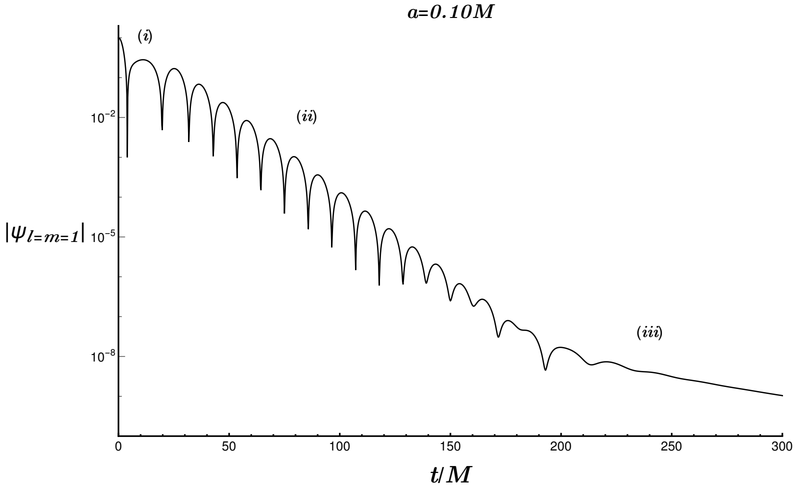
E/J conservation



E/J conservation

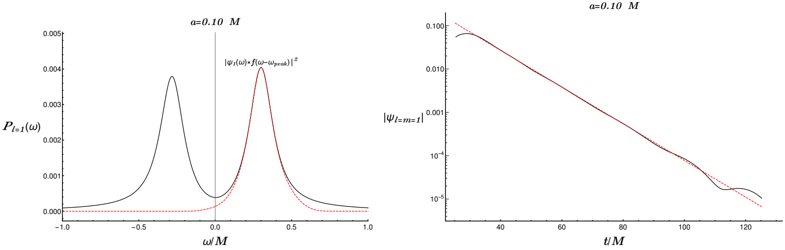


BH response



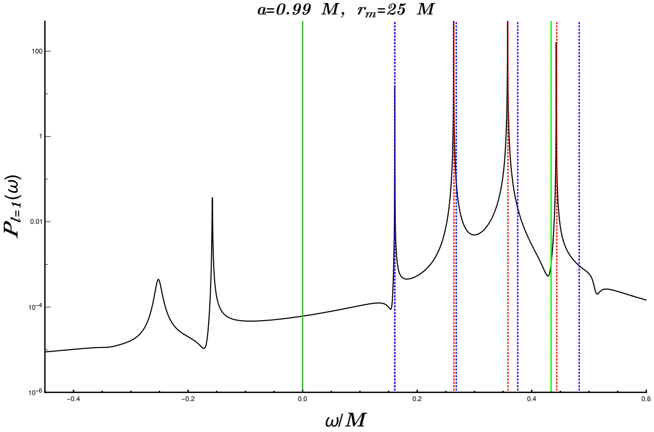
- (i) Prompt-response: depends on the specific initial data
- (ii) Ringdown: quasinormal modes
- (iii) Power-law tail

QNM test



- Peak finding
- Fit with Lorentzian: $\frac{(\pi\nu)^{-1}}{(\omega-\omega_{peak})^2+\nu^2} \to \hat{\omega} = \omega_{peak}$
- Filter: $f(\omega - \hat{\omega}) = \exp(-\frac{(\omega-\hat{\omega})^4}{(nd\omega)^4})$
- linear fit: $\log|\psi| = q + \nu_{peak} \cdot t \to \nu = \nu_{peak}$
- $$\omega_{l=m=1} = 0.301143(\pm 0.03\%) - i0.0974556(\pm 0.09\%)$$

Mirror test

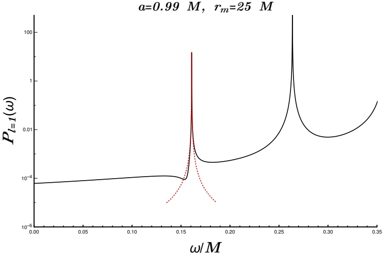


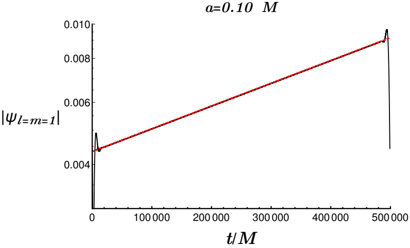
- green: superradiant regime
- red: results from reference 1+1D code[6]
- blue: approximated formula[7] $\omega_{nlm} \simeq \frac{(n+\frac{\ell}{2}+1)}{r_{*,mirror}}$

[6]Dolan (2013)

[7]Cardoso et al., PRD 70 044039 (2004)

Extraction of superradiant modes





$$\omega_{l=m=1} = 0.160877(\pm 0.03\%) + i7.42416 \cdot 10^{-6}(\pm 0.2\%)^8$$

[8] relative errors w.r.t. results of Dolan(2013), in agreement with Cardoso et al.(2004) analytical and numerical results.

Next steps

Spacetime dependent mass terms: $\mu^2 = \mu(r,\theta)^2$

- Accretion disks, plasma-frequency induced mass

$$\omega^2 = k^2 - \omega^2_{plasma}, \quad \omega^2_{plasma} \sim e^2 n_e(r,\theta)$$

- Einstein-Gauss-Bonnet, effective mass term

$$m^2_{GB} = f(\phi)\mathcal{R}_{GB}, \quad \mathcal{R}_{GB} = \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

- Non-trivial self-coupling

$$V(\phi) \sim \frac{g}{3}\phi^3, \frac{\lambda}{4}\phi^4$$

THANK YOU FOR YOUR ATTENTION!

BONUS SLIDES

Perturbation in Kerr metric

$(\nabla^2 - \mu^2)\Psi = 0$, ansatz: $\Psi = \int d\omega e^{-i\omega t + im\phi} \sum_\ell S_{\ell m}(\theta) R_{\ell m}(r)$ [9]

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dS_{\ell m}}{d\theta}) + (a^2(\omega^2-\mu^2)\cos\theta^2 - \frac{m^2}{\sin\theta^2} + A_{\ell m})S_{\ell m} = 0$$

$$\frac{d^2 R_{\ell m}}{dr_*^2} + V_{eff} R_{\ell m} = 0 \tag{8}$$

$$V_{eff}(r) = \frac{(K^2 - \Delta(\lambda + \mu^2 r^2))}{(r^2+a^2)^2} + \beta^2 + \frac{\Delta}{r^2+a^2}\frac{d\beta}{dr} \tag{9}$$

$$K \equiv (r^2+a^2)\omega - am, \qquad \lambda \equiv A_{\ell m} + a^2\omega^2 - 2am\omega, \qquad \beta \equiv \frac{r\Delta}{(r^2+a^2)^2}$$

$\rightarrow$ **Superradiance condition** $\quad \dfrac{\omega - m\Omega_H}{\omega} < 0 \qquad (10)$

[9] Generalized to $s = \pm 1, \pm 2$, Teukolsky master equations

Boundary conditions: Spurious reflections

- Implementation of BC on finite grids: spurious reflection
- $\nu_{\ell,mirror} \sim r_0^{-2(\ell+1)}, \quad r_\infty = 500M$

$$\rightarrow \tau_{boundary} \sim (500M)^4 \sim 6,25 \cdot 10^{10} M \ll (\nu_{l=m=1})^{-1} \sim 10^6 M$$

- Perfectly-matched layers: introduce artificial damping after left boundary
- Spurious reflected flux is damped away

Boundary conditions: Perfectly-mathced layers

$$\ddot{u} = u''$$

First-order rearrangement: $\dot{u} = v'$, $\dot{v} = u'$
Solution: $u \sim \exp -i\omega(t \pm x)$

Introduce x-dependent damping term:

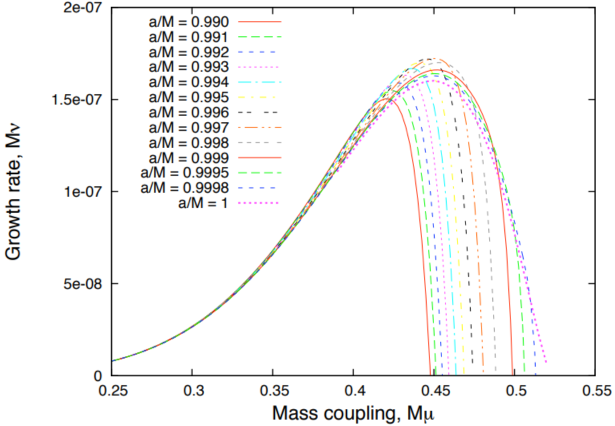
$$\dot{u} = v' - \gamma(x)u$$
$$\dot{v} = u' - \gamma(x)v$$

Solution:

$$u \sim \exp\left(-i\omega(t \pm x) \pm \int_y^x \gamma(y)dy\right)$$

Exponential attenuation for both left and right going waves in PML!

Growth rate - μM



Growth rate - r_m