Probing Regular Black Hole Spacetime with Scalar Field

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Motivation

- Having naked singularity solution or irregular horizon when scalar field is present was predicted already by J. E. Chase in 1970, "Chase Theorem": Any static spherically symmetric vacuum solution minimally coupled to scalar field can not have a regular horizon, if there exists any horizon it would also be the locus of a true singularity.

- Nonlinear Electrodynamics as a good candidate for non-vacuum situation, analysis of gravitating case rather than perturbative approach

J. E. Chase in 1970, Commun. math. Phys, 19, 276
Fields Equations

Lagrangian describing a scalar field minimally coupled to gravity and also Nonlinear Electrodynamics

\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ \mathcal{R} + \nabla_\mu \phi \nabla^\mu \phi + \mathcal{L}(F) \right] \]

- \( \phi \) Real Massless Scalar Field
- \( F = F_{\mu\nu}F^{\mu\nu} \) Electromagnetic Field Invariant

\[ G_{\mu\nu} = SF T_{\mu\nu} + NE T_{\mu\nu} \]
Static Scalar Field

Assuming the static spherically symmetric metric

\[ ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + R(r)^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

The energy momentum tensor

\[ \text{SF} \ T^\mu_\nu = f \frac{\varphi^2_r}{2} \text{diag} \{ -1, 1, -1, -1 \} \]

The wave equations of Radial scalar field

\[ \Box \varphi = 0 \]

where \( \Box \) is a standard d’Alembert operator.

\[ f(r) = 1 \]
\[ R(r) = r^2 - \chi^2 \]
\[ \varphi(r) = \frac{1}{\sqrt{2}} \ln \left\{ \frac{r - \chi}{r + \chi} \right\} \]
Properties of Scalar Field Solution

- $r \to \infty$ the scalar field is vanishing
- Asymptotically flat
- The solution is representing a time-like naked singularity
- Quantization of the spacetime to remove the singularity
- In dynamic case for some parameters there exist event horizon!
Janis, Newman and Winicour Solution

\[ ds^2 = -f(\tilde{R})dt^2 + \frac{1}{f(\tilde{R})} \left\{ d\tilde{R}^2 + (\tilde{R}^2 - M^2)d\Omega^2 \right\} , \]

in which

\[ f(\tilde{R}) = \left[ \frac{\tilde{R} - M}{\tilde{R} + M} \right]^{\frac{1}{\mu}} \]

\[ \phi = \sqrt{(\mu^2 - 1)/2 \ln f(\tilde{R})} \]

- When \( \mu \to \infty \), it would be our solution
- The event horizon is a singular point

Phys. Rev. Lett. 20, 878 (1968);
Introduction to Nonlinear Electrodynamics

The idea of Non-Linear Electrodynamics (NED) is about a century old but it was made popular in 1930s by Born and Infeld. The main goal was solving the point charge singularity:

$$\frac{q}{r^2} \Rightarrow \frac{q}{r^2 + a^2}$$

- Resolve the spacetime singularity $\Rightarrow$ Regular Black Holes
- Wide application in different theories
Different forms of Nonlinear Electrodynamics

- Born-Infeld (BI) theory
- Hoffmann-Born-Infeld (HBI) theory
- Logarithmic Lagrangian
- Power Maxwell (PM)

and many many other models!
The energy momentum tensor

\[ T_{\mu}^{\nu} = \frac{1}{2} \left\{ \delta_{\nu}^{\mu} \mathcal{L} - (F_{\nu\lambda} F^{\mu\lambda}) \mathcal{L}_F \right\} \]

\[ \mathcal{L}_F = \frac{d\mathcal{L}(F)}{dF} \]

The modified Maxwell field equations

\[ \partial_{\mu} (\sqrt{-g} \mathcal{L}_F F^{\mu\nu}) = 0 \]

Electromagnetic Field Invariant

\[ F = F_{\mu\nu} F^{\mu\nu} \]
Born-Infeld

Born and Infeld Lagrangian:

\[ \mathcal{L}_{BI} = 4\beta^2 \left( 1 - \sqrt{1 + \frac{F}{2\beta^2}} \right) \]

- \( \beta \) is BI parameter
- \( \lim_{\beta \to \infty} \mathcal{L}_{BI} = -F \) (Maxwell limit)
- strong field limit \( \to \mathcal{L}_{BI} \sim \sqrt{F} \)
- for Electric Charge \( q_e \), \( F = -\frac{2q_e^4}{r^4 + q_e^4/\beta^2} \Rightarrow "Regular" \)
Regular Black Holes

Generic solution for having RBH

\[ f(r) = 1 - \frac{2C_1 r^{\sigma-1}}{(r^\beta + q)^{\frac{\sigma}{\beta}}} \]

where \( \sigma > 1, \beta > 0 \).

- If \( q = 0 \) the solution is Schwarzschild
- If \( \sigma = 3 \) and \( \beta = 2 \) the solution is Bardeen
- If \( \sigma = 3 \) and \( \beta = 3 \) the solution is Hayward

- \( F \Rightarrow "Singular" \)
- Spacetime and the EMT \( \Rightarrow "Regular" \)
Square Root Model  $\mathcal{L} = -\sqrt{F}$

Maxwell 2-form

$$\mathbf{F} = F_{\theta\phi} \, d\theta \wedge d\phi,$$

where $F_{\theta\phi} = q_m \sin \theta$ and $F = \frac{2q_m^2}{R^4}$

The energy momentum tensor

$$\text{NE } T^\mu_\nu = \text{diag} \left\{ -\frac{\sqrt{F}}{2}, -\frac{\sqrt{F}}{2}, 0, 0 \right\}$$

The corresponding solution is

$$f(r) = \alpha - \frac{2m}{r}$$

$$R(r) = r$$

where $\alpha = 1 - \sqrt{2} \, q_m$
Properties of NE Solution

- \( F = \frac{2q_m^2}{r^4} \Rightarrow \text{"Singular"} \)
- The corresponding metric is not asymptotically flat
- It is a Black Hole solution
- Similar to the solution of geometry outside the core of so-called global monopole
Scalar Field and NE

Metric functions

\[ f(r) = \frac{C_0}{\sqrt{2} \chi} \]
\[ R(r) = r^2 - \chi^2 \]

Constraint Eq: \( G^t_t - (\text{NE} T^t_t + \text{SF} T^t_t) = 0, \)

\[
f \left( \frac{R,r}{R} \right)^2 + \frac{R,r}{R} f_{,r} - \frac{1}{R^2} - f \frac{R,rr}{R} + \frac{q_m}{\sqrt{2}} \frac{1}{R^2} = 0
\]

which leads to

\[ q_m = \frac{C_0}{\chi} - \sqrt{2} \]
Properties: Scalar Field and NE

- If $f = 1$ then $q_m = 0$
- **No Horizon** unless scalar field vanishes identically
- Kretschmann scalar is diverging at $r = \chi$
Different Model of NE

Scalar Field + Regular Models (magnetic charge)

\{Bardeen, Hayward, Generic Model\}

⇒ NOT EVEN BLACK HOLE SOLUTION

Long Eq.s

Not Explicit Solu.

Several constraints
No Black hole solutions

Different coordinates:

\[ ds^2 = -f(r) \, dt^2 + \frac{h(r)}{f(r)} \, dr^2 + r^2 \, d\Omega^2 \]

Solution:

\[ f = \frac{f_0 \, h}{\sqrt{r^3 \, h, r}} \]

\[ \sqrt{-g} = \sqrt{h} \, r^2 \, \sin \theta \]

Ricci Scalar:

\[
Ricci = \frac{2}{r^2} + \frac{f_0}{4 \, \sqrt{r^5 \, h, r}} \left\{ 2 \, r \left( \frac{h, r}{h} \right)^2 + \frac{h, r - r \, h, rr}{h} \right. \\
+ \frac{1}{r^3 \, (h, r)^2} \left[ r^2 \left( 2(h, rrr) \, h, r - 3(h, rr)^2 \right) + (r \, h, r^2), r \right] \right\}
\]
Assumptions:

- having horizon at \( r = r_0 \) \( \rightarrow \) \( h(r_0) = 0 \)
- \( h_r(r_0) \) is finite and nonzero

Results:

- Vanishing \( h \) means diverging curvature scalars
- Highlighted terms would be zero:

\[
h = -\frac{h_0}{r^2 + h_1} \rightarrow \left\{
\begin{align*}
\text{Imaginary metric} \\
f &\sim 1/r^2
\end{align*}
\right.
\]
Wave equation for Black hole solutions close to event horizon

Scalar perturbation obeying the Klein-Gordon equation,

\[ \Box \psi(t, r, \theta, \phi) = 0 \]

for the Static Spherical Symmetric (with \( R = r \))

\[ f \psi_{,rr} + \left( f, r + \frac{2f}{r} \right) \psi_{,r} - \frac{\psi_{,tt}}{f} + \frac{1}{r^2} \left\{ \psi_{,\theta\theta} + \cot \theta \psi_{,\theta} + \frac{\psi_{,\phi\phi}}{\sin^2 \theta} \right\} = 0 \]
By applying separation variables

$$\Psi(t, r, \theta, \phi) = e^{-i\omega t} \frac{\psi(r)}{r} Y^m_l(\theta, \phi)$$

Radial equation is

$$f \psi_{,rr} + f_r \psi_{,r} - \left( \frac{l(l + 1)}{r^2} - \frac{\omega^2}{f} + \frac{f_r}{r} \right) \psi = 0$$

Since we are interested in perturbation around horizon, we assume

$$f = A(r - r_0) + O((r - r_0)^2)$$
In the end

\[ \psi = (r - r_0) \frac{-l \omega}{A} \left[ \psi_0 r^{n_1} F_1 \left( a_1, b_1; n_1; \frac{r}{r_0} \right) + \psi_1 r^{n_2} F_1 \left( a_2, b_2; n_2; \frac{r}{r_0} \right) \right] \]

- \( (r - r_0) \frac{-l \omega}{A} \) is the dominant term, so \( \psi, r \sim (r - r_0)^{-1} \)
- diverging stress energy momentum tensor of the scalar field
- generic test scalar field energy momentum tensor blows up on the horizon
Conclusions

- Chase theorem applies for NE sources
- Test scalar field EMT blows up generally on event horizon
- Gravitating scalar fields with NE produce singular horizons
- Explicit solution for square root Lagrangian mimicks global monopole with modified singularity
THANK YOU