Causal set theory and quantum fields

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Outline

• Introduction to causal sets
• Causal sets and scalar fields
• Entanglement entropy
• Summary and comments
A causal set (causet) is a locally finite ordered set.

It is a pair $(C, \preceq)$, given by a set $C$ and a partial order relation $\preceq$ that is:

- Reflexive \[ \forall x \in C: x \preceq x \]
- Antisymmetric \[ \forall x, y \in C: x \preceq y \preceq x \Rightarrow x = y \]
- Transitive \[ \forall x, y, z \in C: x \preceq y \preceq z \Rightarrow x \preceq z \]
- Locally finite \[ \forall x, y \in \text{card}\{z \in C: x \preceq z \preceq y\} < \infty \]

Causal structure gives $9/10$ of the metric. Hawking, King, McCarthey, Malament, Levichev

The rest (a conformal factor) can be fixed by volume information.

In a causet the volume is naturally associated with the number of elements $N$

\[
\text{VOLUME + ORDER} = \text{GEOMETRY} \Rightarrow \text{NUMBER + ORDER} = \text{GEOMETRY}
\]
\[ L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

- Density \( \rho \propto \ell^{-1/d} \)
- \( \langle N \rangle = \rho V \)
- Spacetime volume \( V_{st} \propto N \)
- Nonlocality
- Fixed background, no dynamics!
A sprinkling of Minkowski st cannot determine a rest frame, an arrow of time, brake translation symmetry or Lorentz symmetry by endowing spacetime with a distinguished lattice.

**Entanglement Entropy**

- Black hole entropy

The study of entanglement entropy in QFT is the study of Planck scale physics

\[
\rho_B = \text{Tr}_A \rho_{\Sigma} \\
S_B = -\text{Tr} \rho_B \log \rho_B \\
S_A = S_B = \infty
\]

Correlations at arbitrarily small scales.

\[S_{A,B} \propto \frac{\text{Area}(\partial A)}{\ell^{d-2}}\]
• Area law from QG-inspired models:
  • Hard spatial cutoff;
  • Lorentz violating theories \( (\partial_t + f(\partial_x))\phi = 0 \), D. Nesterov, S.N. Solodukhin, Nucl.Phys. B842 (2011) 141-171;
  • Nonlocal scalar fields \( f(\Box)\phi = 0 \) (continuum limit of CS d’Alembertian, string-field theory)
    D.M.T. Belenchia, A. Benincasa, ML, S. Liberati, Class.Quant.Grav. 35 (2018) no.7, 074002
  • New examples? ML, J. Pye (work in progress)
Free Scalar Fields and Entanglement Entropy

A causal set is intrinsically spacetime discrete without the analogue of a Cauchy hypersurface where to define states and initial data.

- New algorithm to compute EE
- New method to quantize (free) fields
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**Spacetime entropy** R.D. Sorkin, J.Phys.Conf.Ser. 484 (2014) 012004

- $i\Delta = G^R - (G^R)^T$ Pauli-Jordan, $\Delta = [\phi, \phi]$  
- $W = \langle \phi \phi \rangle$ Wightman,  
  $\Delta = 2 \Im(W); \Re(W)$ specifies the vacuum.

$$W \cdot \nu = i\lambda \Delta \cdot \nu \quad \Delta \nu \neq 0 \rightarrow S = \sum \lambda \log |\lambda|$$


Cannot split positive/negative frequencies.

$$\Box \rightarrow G^R \rightarrow \Delta \rightarrow [\phi, \phi] \quad \text{freq} \rightarrow a \rightarrow |0\rangle \rightarrow W$$

- $G^R \rightarrow \Delta \rightarrow W_{SJ}$
  - $W - W^* = i\Delta$
  - $W \geq 0$
  - $WW^* = 0$

$W_{SJ} = \text{Pos}(i\Delta)$

$W$ specifies completely the free theory.

\[ G_{xy}^{(2)} = \frac{1}{2} C_{xy} \]

\[ G_{xy}^{(3)} = \frac{1}{2\pi} \left( \frac{\pi \rho}{12} \right)^{1/3} ((C + I)^2)_{xy}^{-1/3} \]

\[ G_{xy}^{(4)} = \frac{\sqrt{\rho}}{2\pi \sqrt{6}} L_{xy} \]

Causal set d’Alembertian operators.

\[ B_{\rho}^{(d)} \phi(x) = (\epsilon \rho)^{2/d} \left( a\phi(x) + \sum_{n=0}^{\infty} b_n \sum_{y \in I_n(x)} \phi(y) \right) \]

\[ \epsilon = \left( \frac{\ell}{l_k} \right)^d, \quad l_k \geq \ell \]

R.D. Sorkin, In *Oriti, D. (ed.): Approaches to quantum gravity* 26-43
L. Glaser, Class.Quant.Grav. 31 (2014) 095007
Free Scalar Fields and Entanglement Entropy

\[ \Delta, \ W_{SJ} = \text{Pos}(i\Delta), \]

\[ W_{SJ}\bigg|_U \ v = i\lambda \Delta\bigg|_U \ v, \quad S = \sum \lambda \log |\lambda|, \quad \ell \propto \rho^{-1/d} \]

\[ N \propto V_{st} \Rightarrow S(U) \propto V_{st} \]
Free Scalar Fields and Entanglement Entropy

\[ W \bigg|_U v = i \lambda \Delta \bigg|_U v, \quad \Delta \bigg|_U v \neq 0 \]

Enlarge \( \ker \Delta \) by imposing a cutoff \( \rightarrow k_{\text{max}} \propto \ell^{-1} = \rho^{1/d} \)

\[ \alpha^{\text{min}} = c_d V^{2d-1} N^{1-1/d} \]

\[ S \propto \frac{A}{\ell^{d-2}} \]
What is causing the spacetime volume law?

- **Nonlocality**
  Nearest neighbours can be arbitrarily far in any given reference frame.
  \[ \rightarrow \text{not localized close to the boundary.} \]

- **Equations of motion**
  In the continuum, it can be proven that
  \[ \text{Im}(\Delta) = \ker(\Box). \]

Take a vector \( v_j \) in the kernel \( \sum \Delta^{ij} v_j = 0 \), then \( v_j \phi^i = 0 \rightarrow \text{eom!} \)

In a causet, the kernel is very small \( \rightarrow \dim \ker \Delta = k \sim O(\log N) \ll N \) in 2d \( \rightarrow \) highly underdetermined system.

The cutoff gives better equations of motion by enlarging the kernel.
Remember: \( \Delta = [\phi, \phi] \) becomes a fully populated matrix! \( \rightarrow \) causality violations!
Summary and Comments

• Entanglement Entropy:
  • Black hole entropy
  • Gravity from EE
  • UV structure of spacetime

• Causal sets and scalar fields:
  • Spacetime volume law!
  • What is its origin?
  • Should we and can we recover the area law?
  • What does it tell us about constructing a QFT in a causal set?

• What else can we do with scalar fields on causal sets?
  • Phenomenology in the continuum approximation, A Belenchia, DMT Benincasa, S Liberati, JHEP 2015 (3), 36
  • Spectral geometry, Y. Yazdi, A. Kempf, Class.Quant.Grav. 34 (2017) no.9, 094001; A. Kempf, ML, Y. Yazdi (in preparation)