

Protostellar disc theory

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“F. Lucchin” PhD School, Bertinoro, October 2026

OUTLINE

- Fundamentals of disc accretion
- Basic equations and simple analytical solutions
- Viscosity and turbulent transport
- Disc-planet interaction
- Dust dynamics in protostellar discs

ACCRETION DISCS IN ASTROPHYSICS

- Accretion discs play a fundamental role in several different contexts in astrophysics
 - central engine in AGN
 - compact objects in galactic binary systems (X-ray binaries, Dwarf novae, etc.)
 - Protostellar discs
- Protostellar discs offer an incredible advantage: we can actually image them at high resolution!

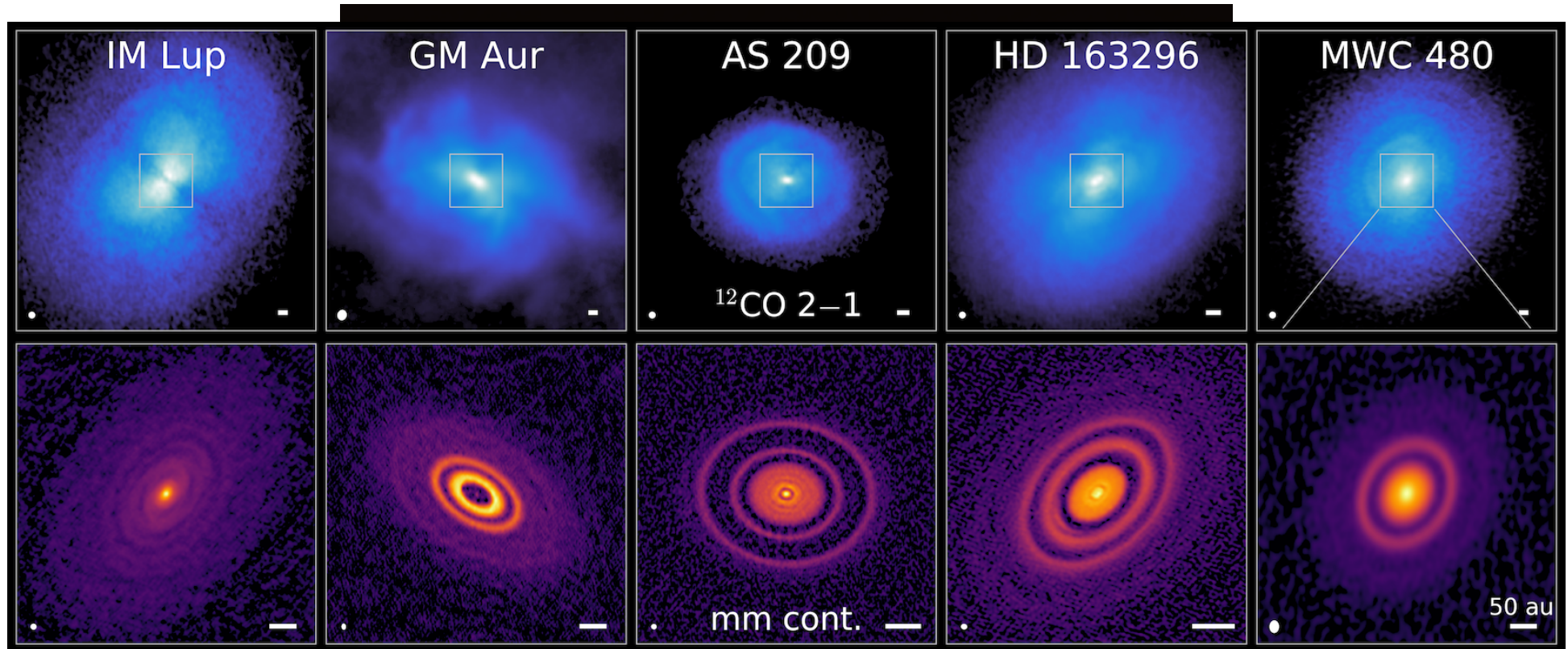
MAIN PROTOSTELLAR DISC OBSERVABLES

ACCRETION DISC IMAGING



Accretion disc in the nucleus of the Active Galaxy M87, seen from the EHT

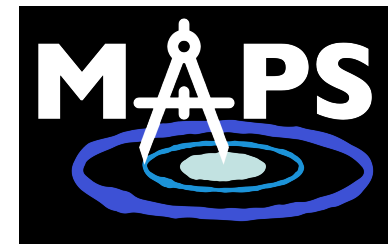
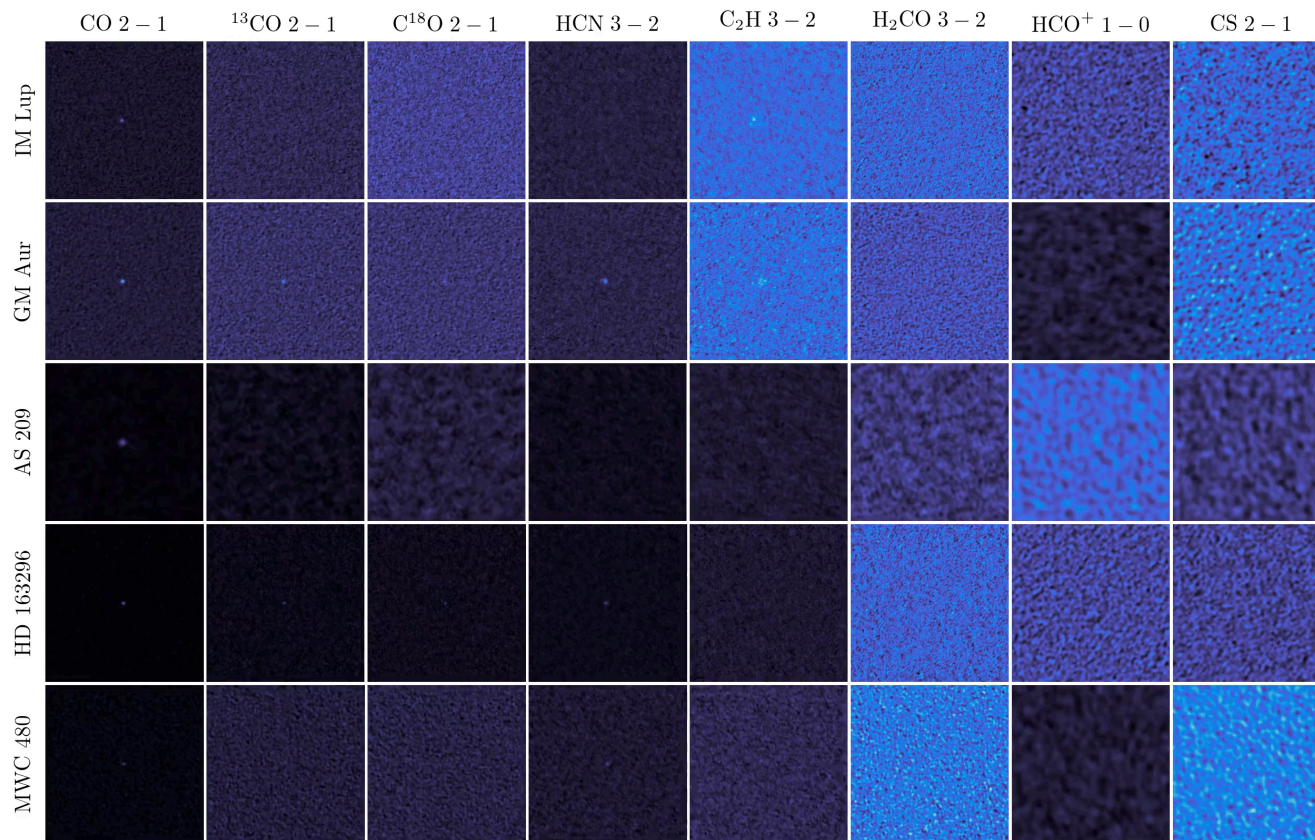
ACCRETION DISC IMAGING



Gas and dust emission from protostellar
accretion discs from ALMA

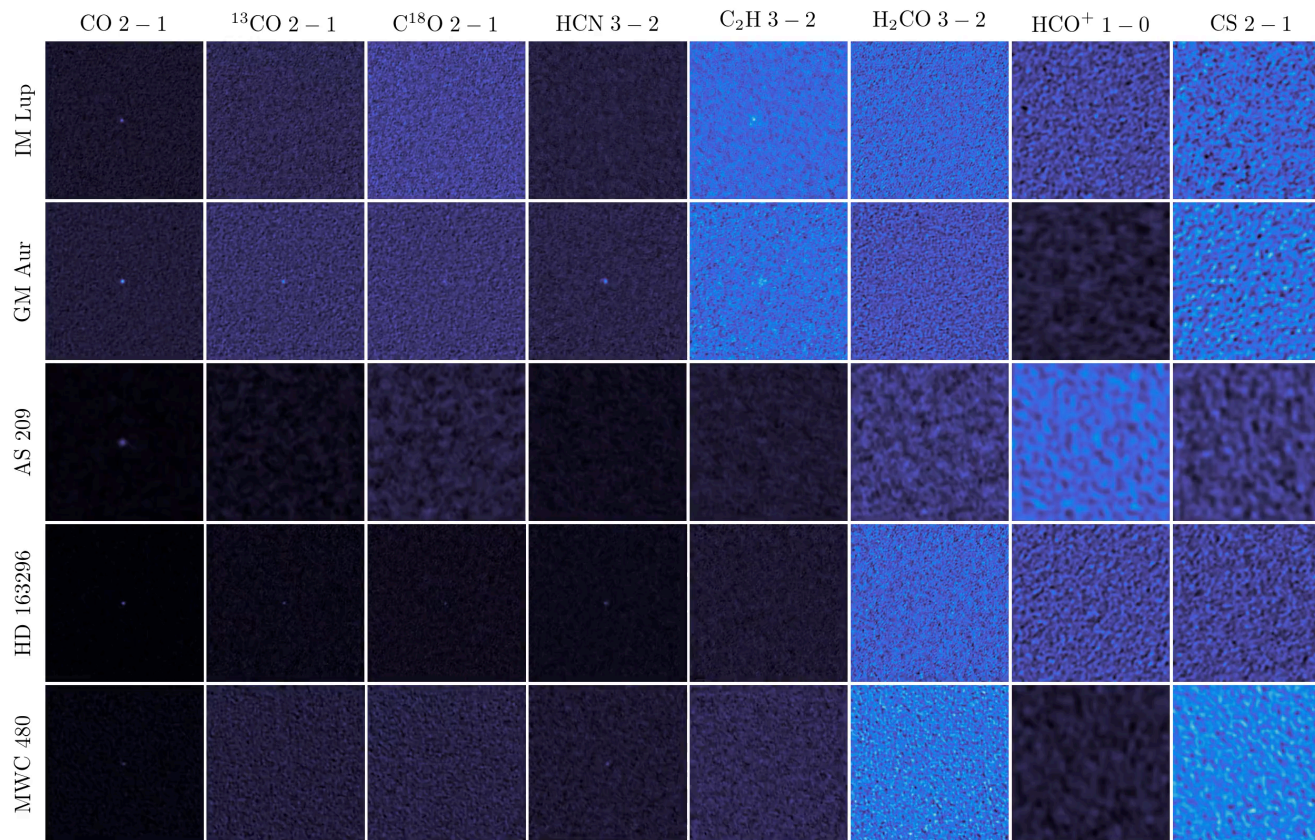
DISC KINEMATICS

- Measuring the (3D) velocity field of the disc gas from Doppler shifted line emission



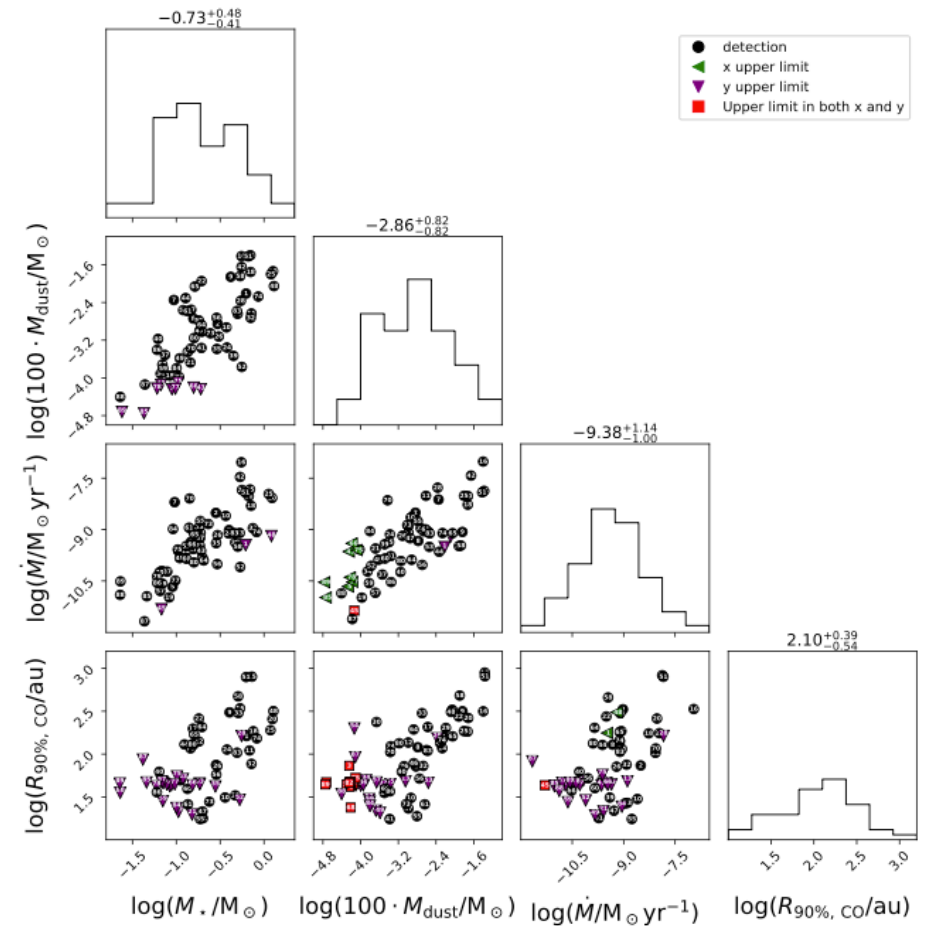
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FUNDAMENTAL DISC PROPERTIES

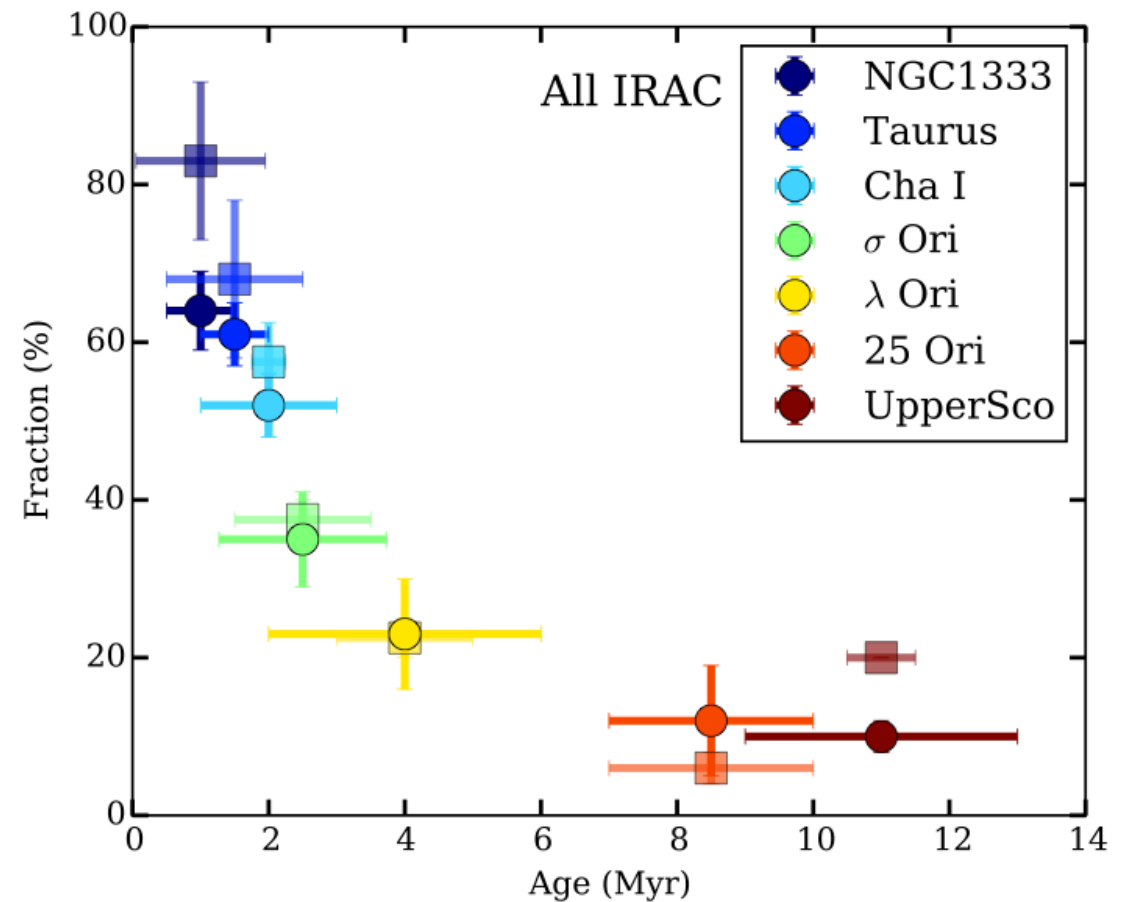
- Wide range of properties, often correlated with each other
- Disc masses: $10^{-4}M_{\odot} - 0.1M_{\odot}$
- Accretion rates: $10^{-11} - 10^{-7}M_{\odot}/\text{yr}$
- Disc sizes: $10 - 10^3\text{au}$
- Disc lifetime: 2-3 Myr



Malanga et al 2026

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Ribas et al 2015

DISC ACCRETION

- Main issue: the removal of angular momentum
- Centrifugal barrier prevents matter from falling into the central object
- Angular momentum disposal:
 - Transport by “viscosity” (Shakura & Sunyaev, 1973)
 - Removal by winds or outflows (Blandford & Payne, 1983 - see lecture by Hardcastle)

GAS DYNAMICS IN DISCS

BASIC EQUATIONS

- Standard fluid equations: continuity, Navier-Stokes, Poisson...

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma v_R) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Phi + \vec{\nabla} \cdot \vec{\sigma}$$

$$P = P(\rho) = c_s^2 \rho$$

$$\sigma_{R\phi} = \eta R \frac{d\Omega}{dR} = \nu \rho R \frac{d\Omega}{dR}$$

$$\Phi_{\star} = -\frac{GM}{r} = -\frac{GM}{(R^2 + z^2)^{1/2}}$$

$$\nabla^2 \Phi_{\text{sg}} = 4\pi G \rho$$

VERTICAL DIRECTION: HYDROSTATIC BALANCE

$$\cancel{\frac{\partial \vec{v}}{\partial t}} + \vec{v} \cdot \cancel{\nabla} \vec{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi + \cancel{\nabla \cdot \vec{\sigma}}$$

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$$c_s^2 \frac{d \ln \rho}{dz} = -\frac{d\Phi}{dz} = -\frac{GMz}{(R^2 + z^2)^{3/2}} \approx -\frac{GMz}{R^3}$$

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$$v_K^2 = GM/R$$

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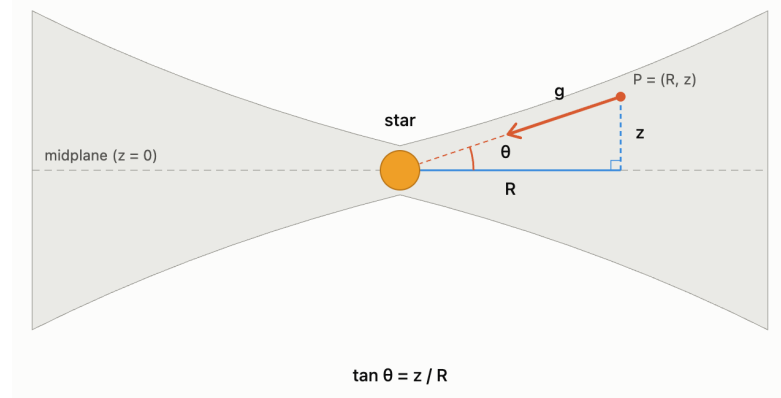
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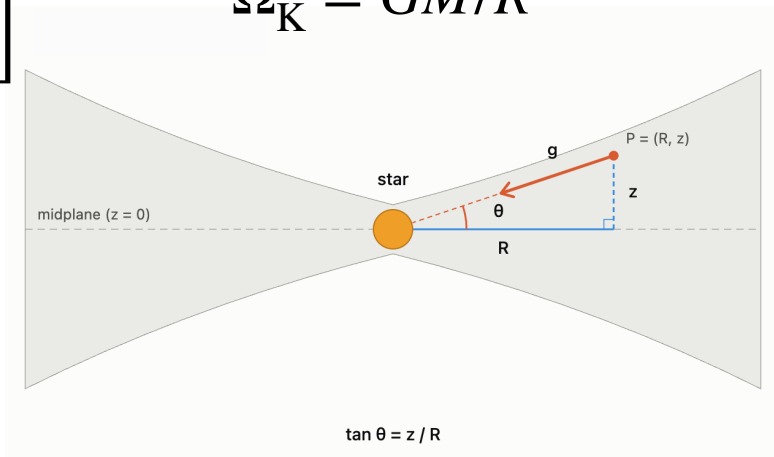
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$$v_K^2 = GM/R$$

$$\ln(\rho/\rho_c) = \frac{1}{c_s^2} \left[\frac{GM}{\sqrt{R^2 + z^2}} - \frac{GM}{R} \right] = \frac{v_K^2}{c_s^2} \left[\frac{1}{\sqrt{1 + z^2/R^2}} - 1 \right]$$

$$\Omega_K^2 = GM/R^3$$



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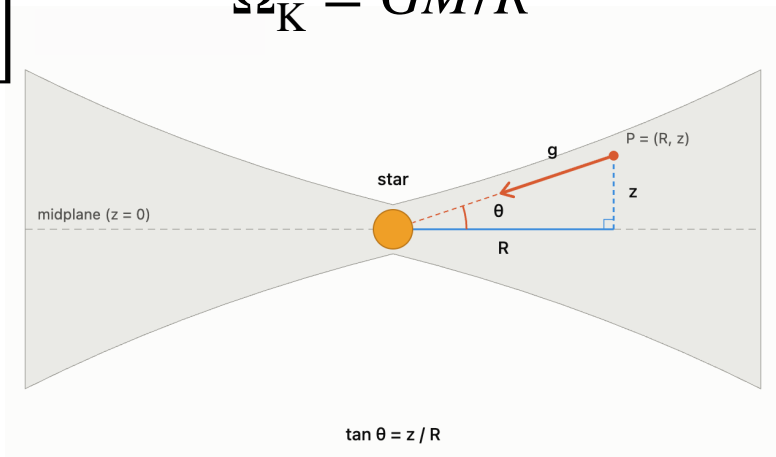
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$$\Omega_K^2 = GM/R^3$$

$$\rho = \rho_c \exp \left[-\frac{z^2}{2H^2} \right]$$

$$H = c_s/\Omega_K$$

$$H/R = c_s/v_K$$



VERTICAL DIRECTION: HYDROSTATIC BALANCE

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- For a self-gravitating disc (Spitzer, 1942)

$$\frac{c_s^2}{\rho} \frac{d\rho}{dz} = -4\pi G \int_0^z \rho(z') dz'$$

$$\rho = \frac{\rho_c}{\cosh^2(z/H)}$$

$$H = \frac{c_s^2}{\pi G \Sigma}$$

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Exercise 1: demonstrate this solution

RADIAL DIRECTION: CENTRIFUGAL BALANCE

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$$-\frac{v_{\phi}^2}{R} = -\frac{1}{\rho} \frac{dP}{dR} - \frac{d\Phi}{dR}$$

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To zero-th order, disc is Keplerian

$$v_{\phi}^2 = R \frac{d\Phi}{dR} = \frac{GMR^2}{(R^2 + z^2)^{3/2}} \approx \frac{GM}{R} = v_K^2$$

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$$P_c(R) = c_s^2(R) \rho_c(R) \propto R^{-\gamma'}$$

Locally isothermal disc

$$T_c(R) \propto c_s^2 \propto R^{-q}$$

RADIAL DIRECTION: CENTRIFUGAL BALANCE

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$$v_{\phi}^2 = v_K^2 \frac{1}{(1 + z^2/R^2)^{3/2}} + c_s^2 \frac{d \ln P}{d \ln R} = v_K^2 \left[\frac{1}{(1 + z^2/R^2)^{3/2}} + \left(\frac{H}{R} \right)^2 \frac{d \ln P}{d \ln R} \right]$$

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In the midplane ($z = 0$)

$$v_{\phi}^2 = v_K^2 \left[1 - \gamma' \left(\frac{H}{R} \right)^2 \right]$$

Out of the midplane ($z \neq 0$)

$$v_{\phi}^2 = v_K^2 \left[1 - \gamma' \left(\frac{H}{R} \right)^2 - q \left(1 - \frac{1}{\sqrt{1 + z^2/R^2}} \right) \right]$$

Radially isothermal discs ($q = 0$) rotate on cylinders, while in general there is a vertical shear

RADIAL DIRECTION: CENTRIFUGAL BALANCE

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Exercise 2: demonstrate the formula for $v_{\phi}(R, z)$

EXERCISE 2

1. Demonstrate that a radially and vertically isothermal disc rotates on cylinders (Poincarè-Wavre theorem)

- Hints:
 - Demonstrate that the pressure term at $z>0$ gives a super-Keplerian rotation term
 - Compute the pressure correction based on the usual Gaussian profile
 - Compute the full azimuthal velocity

$$v_{\phi}^2 = v_K^2 \left[1 - \gamma' \left(\frac{H}{R} \right)^2 - q \left(1 - \frac{1}{\sqrt{1 + z^2/R^2}} \right) \right]$$

2. Compute the effect of an exponential truncation to the disc on rotation velocity

AZIMUTHAL DIRECTION: ENTERS VISCOSITY

Azimuthal component of Navier-Stokes equation can be written in terms of angular momentum conservation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = - \frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \Phi + \vec{\nabla} \cdot \vec{\sigma}$$

$$\frac{\partial}{\partial t} \left(\Sigma R v_\phi \right) + \frac{1}{R} \frac{\partial}{\partial R} \left(R v_r \Sigma R v_\phi \right) = \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \Omega' \right)$$

Note: viscous torque on the RHS is a surface term → it only transports angular momentum, without removing it

MASTER EQUATION FOR Σ

- Combining continuity and angular momentum conservation

$$\frac{\partial}{\partial t} \left(\Sigma R v_\phi \right) + \frac{1}{R} \frac{\partial}{\partial R} \left(R v_r \Sigma R v_\phi \right) = \frac{1}{R} \frac{\partial}{\partial R} \left(\nu \Sigma R^3 \Omega' \right)$$

+

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left(R \Sigma v_R \right) = 0$$

=

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left(R^{1/2} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right) \right)$$

MASTER EQUATION FOR Σ

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R} \frac{\partial}{\partial R} \left(R^{1/2} \frac{\partial}{\partial R} (\nu \Sigma R^{1/2}) \right)$$

- A diffusion equation for Σ
- Typical evolutionary timescale $t_\nu = R^2/\nu$ - the *viscous timescale*

$$\frac{t_\nu}{t_{\text{dyn}}} = t_\nu \Omega = \frac{\Omega R^2}{\nu} = Re$$

STEADY STATE SOLUTIONS

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (R \Sigma v_R) = 0 \quad \longrightarrow \quad \dot{M} = -2\pi R v_R \Sigma = \text{const}$$

$$\frac{\partial}{\partial t} (\Sigma R v_\phi) + \frac{1}{R} \frac{\partial}{\partial R} (R v_r \Sigma R v_\phi) = \frac{1}{R} \frac{\partial}{\partial R} (\nu \Sigma R^3 \Omega')$$

$$\dot{J} = \dot{M} R^2 \Omega + 2\pi \nu \Sigma R^3 \Omega' = \dot{M} R^2 \Omega - 3\pi \nu \Sigma R^2 \Omega = \dot{M} R_{\text{in}}^2 \Omega_{\text{in}}$$

$$3\pi \nu \Sigma = \dot{M} \left[1 - \sqrt{\frac{R_{\text{in}}}{R}} \right]$$

TIME DEPENDENT SOLUTIONS

1. The spreading ring - Lynden-Bell & Pringle (1974)

Mon. Not. R. astr. Soc. (1974) **168**, 603–637.

THE EVOLUTION OF VISCOUS DISCS AND THE ORIGIN OF THE NEBULAR VARIABLES

D. Lynden-Bell and J. E. Pringle

(Received 1974 March 18)

SUMMARY

The evolution of discs under the action of viscosity is studied by both similarity solutions and Green's functions. The angular momentum is steadily concentrated onto a small fraction of the mass which orbits at greater and greater radii while the rest is accreted onto the central body.

We assume that the angular momentum excess of a proto-star is initially concentrated onto one-third of the total mass which forms a disc orbiting the new-born star. Viscous dissipation in this disc will cause it to shine with a luminosity greater than the final main sequence star for a period of 10^5 yr or so. Most of the properties of T Tauri stars can be explained as a consequence of disc evolution. Flares in Flare stars are interpreted as the entry of blobs of an old disc into the late type stellar atmospheres. On this hypothesis flaring activity could be observed in M stars of up to 5×10^8 yr old, and planetary systems will be common. Disc solutions appropriate to dwarf novae and X-ray sources are also given.

TIME DEPENDENT SOLUTIONS

1. The spreading ring - Lynden-Bell & Pringle (1974)

- Analytical solution for the case where $\nu = \text{const}$
- Initial condition: $\Sigma(R,0) = \frac{m}{2\pi R_0} \delta(R - R_0)$

$$\Sigma(R, t) = \frac{m}{\pi R_0^2} \frac{x^{-1/4}}{\tau} \exp \left[-\frac{1 + x^2}{\tau} \right] I_{1/4} \left(\frac{2x}{\tau} \right)$$

$$x = R/R_0$$

$$\tau = 12\nu t/R_0^2$$

TIME DEPENDENT SOLUTIONS

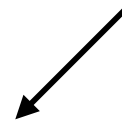
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Modified Bessel function



$$x = R/R_0$$

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TIME DEPENDENT SOLUTIONS

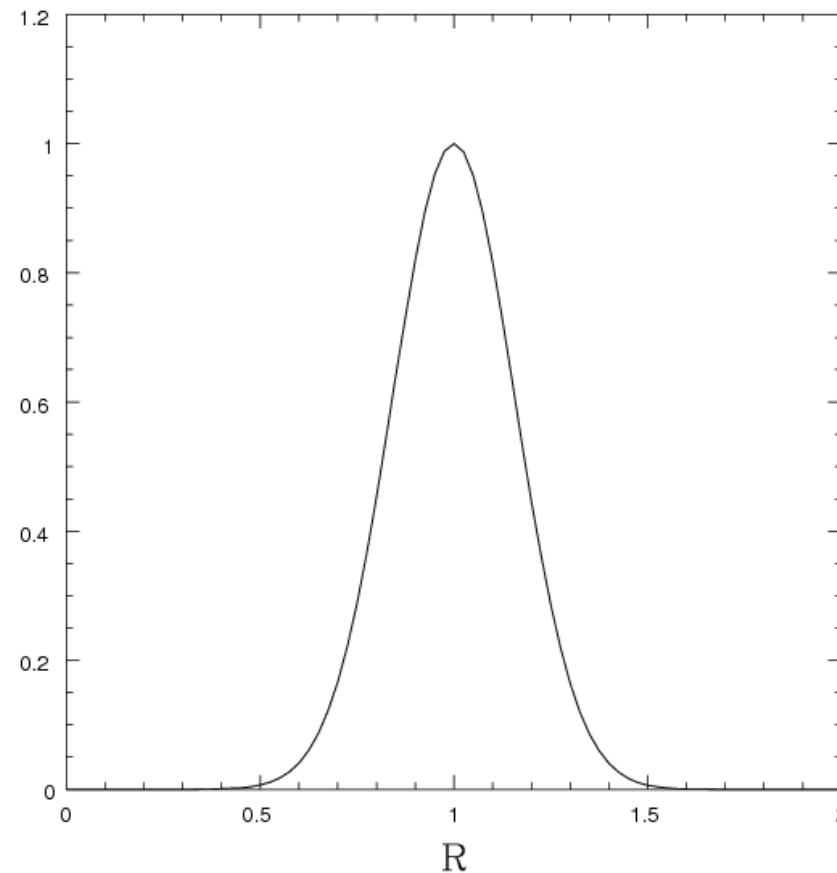
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$$\Sigma(R, t)$$

$$x = R/R_0$$

$$\tau = 12\nu t/R_0^2$$



Modified Bessel function

/

$$\frac{2x}{\tau}$$

TIME DEPENDENT SOLUTIONS

2. The self-similar solution - Lynden-Bell & Pringle (1974)

- Analytical solution for the case where $\nu = \nu_c (R/R_c)^\gamma$
- Initial condition:

$$\Sigma(R,0) = \frac{M_0}{2\pi R_c^2} (2 - \gamma) \left(\frac{R}{R_c} \right)^{-\gamma} \exp \left[- \left(\frac{R}{R_c} \right)^{(2-\gamma)} \right]$$

$$\Sigma(R, t) = \frac{M_0}{2\pi R_c^2} (2 - \gamma) \left(\frac{R}{R_c} \right)^{-\gamma} T^{-\eta} \exp \left(- \frac{(R/R_c)^{(2-\gamma)}}{T} \right)$$

$$T = 1 + t/t_\nu$$

$$t_\nu = \frac{R_c^2}{3(2 - \gamma)\nu(R_c)}$$

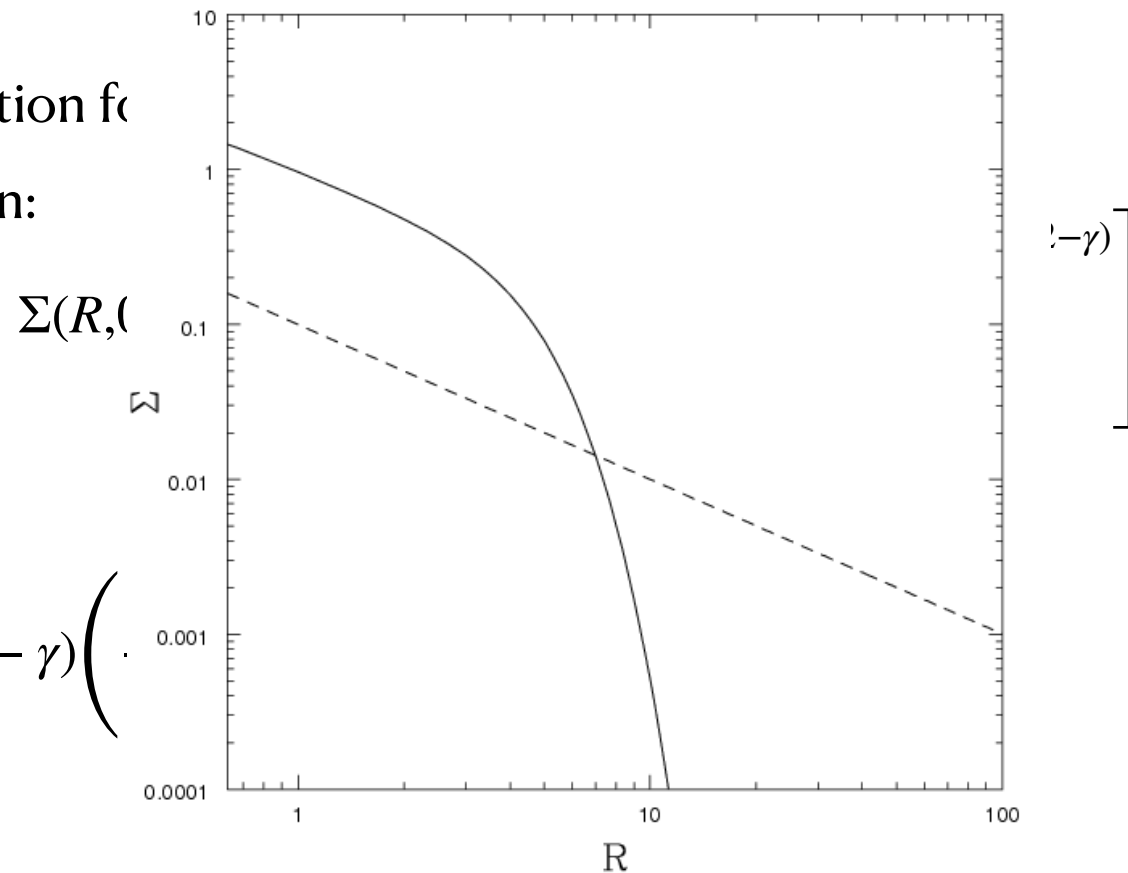
$$\eta = \frac{5/2 - \gamma}{2 - \gamma}$$

TIME DEPENDENT SOLUTIONS

2. The self-similar solution - Lynden-Bell & Pringle (1974)

- Analytical solution for $\Sigma(R, t)$
- Initial condition:

$$\Sigma(R, t) = \frac{M_0}{2\pi R_c^2} (2 - \gamma) \left(\frac{R}{R_c} \right)^{-(2-\gamma)} \left(1 - \left(\frac{R}{R_c} \right)^{-(2-\gamma)} \right)$$



$$T = 1 + t/t_\nu$$

$$t_\nu = \frac{R_c^2}{3(2 - \gamma)\nu(R_c)}$$

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TIME DEPENDENT SOLUTIONS

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$$M_d(t) = M_0 T^{(1-\eta)}$$

$$\dot{M} = -\frac{dM_d}{dt} = (\eta - 1) \frac{M_0}{t_v} T^{-\eta}$$

$$t_{\text{disc}} = \frac{M_d(t)}{\dot{M}_{\text{acc}}(t)} = 2(2 - \gamma)(t + t_v)$$

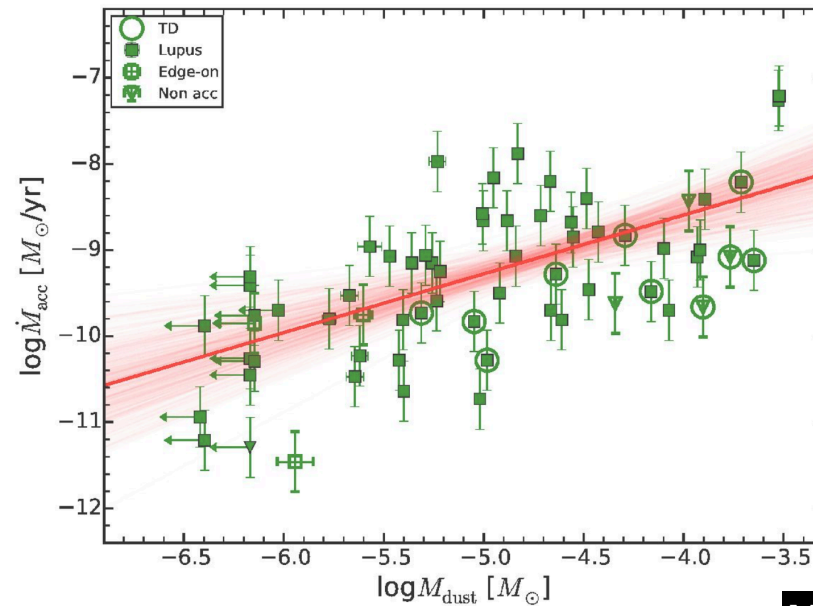
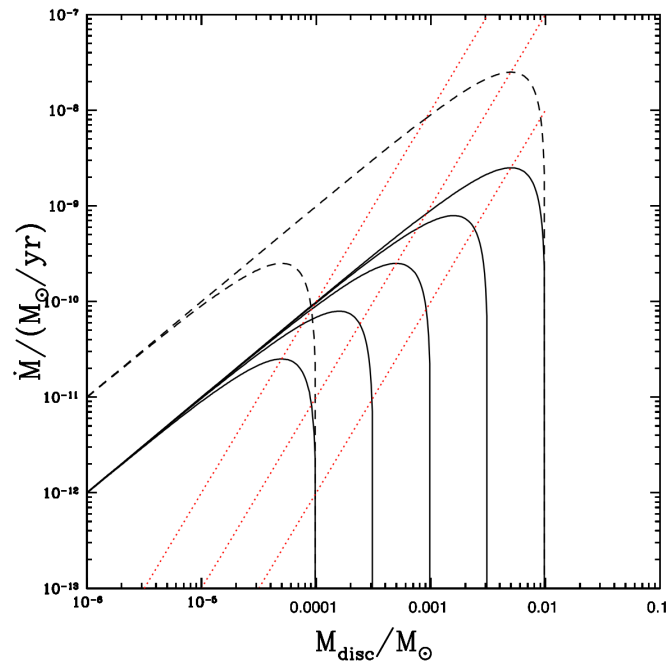
Disc lifetime increases as time goes by

DISC ISOCHRONES FOR THE SELF-SIMILAR SOLUTION

Lodato et al (2017)

- Where does a population of discs **with the same age** stand in the $\dot{M} - M_d$ plane?

$$\dot{M} = \frac{M_d}{2(2 - \gamma)t} \left[1 - \left(\frac{M_d}{M_0} \right)^{2(2-\gamma)} \right]$$



Manara et al 2016

EXERCISE 3

- A malevolent deity suddenly removes the outer parts of a disc, reducing its outer radius from R_1 to $R_2 < R_1$
- What happens to the disc lifetime?
 - A. The viscous timescale at R_2 is smaller than the one at R_1 : the lifetime is reduced
 - B. The viscous timescale is a local property and does not depend on whether there is or not an outer disc: the lifetime stays the same
 - C. After truncation, the disc expands beyond R_2 where the viscous timescale is longer: the lifetime is increased

GAS THERMODYNAMICS

DISC THERMAL STRUCTURE

- For high $\dot{M}$, disc is heated by the accretion process, for low $\dot{M}$ is heated by irradiation

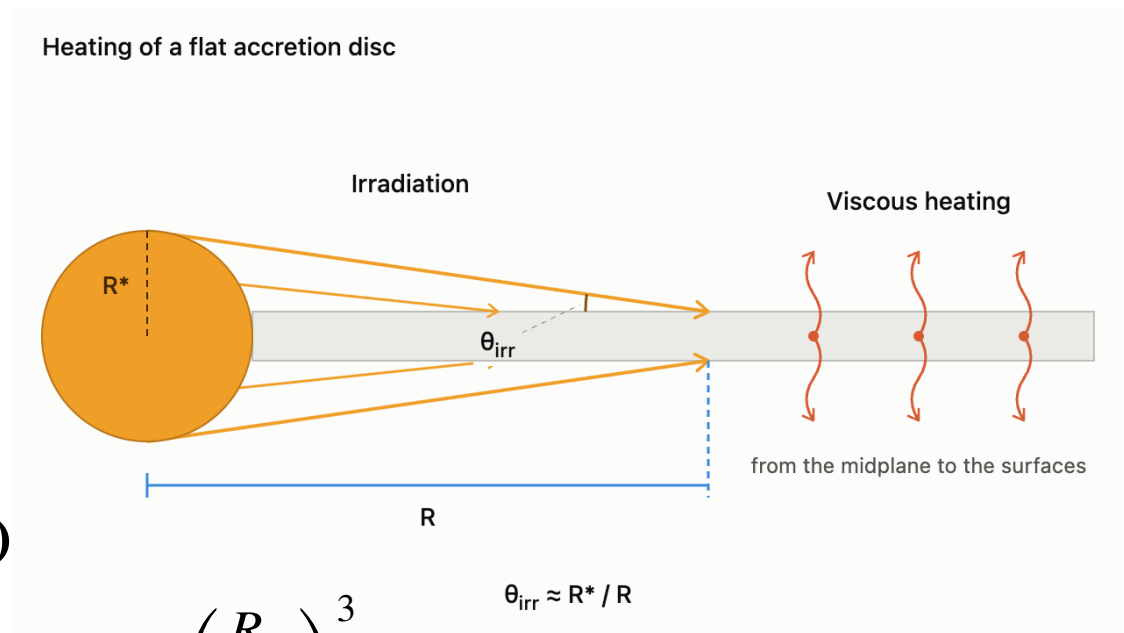
- Active discs (internally heated)**

$$2\sigma_{\text{SB}}T_s^4 = D(R) = \nu\Sigma(R\Omega')^2 \approx \frac{3}{4\pi} \frac{GM\dot{M}}{R^3}$$

- Passive discs (externally heated)**

$$\sigma_{\text{SB}}T_s^4 = \frac{L_\star}{4\pi R^2} \sin \theta_{\text{irr}} \approx \frac{1}{4\pi} \frac{\pi\sigma_{\text{SB}}T_\star^4 R_\star^2}{R^2} \frac{R_\star}{R} \approx f\sigma_{\text{SB}}T_\star^4 \left(\frac{R_\star}{R} \right)^3$$

Note: in both cases $T_s \propto R^{-3/4}$



DISC THERMAL STRUCTURE

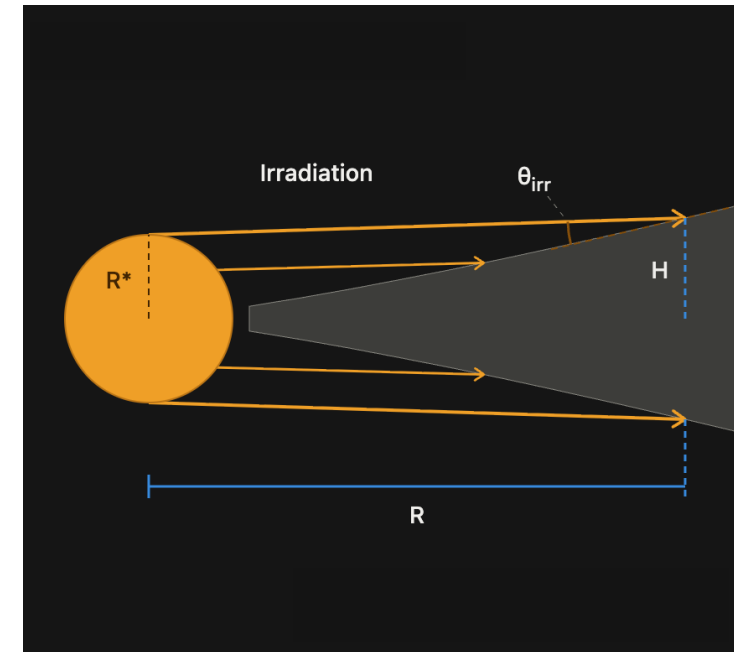
- For high $\dot{M}$, disc is heated by the accretion process, for low $\dot{M}$ is heated by irradiation

- Irradiated flared disc** $\frac{H}{R} \propto R^\beta$

$$\theta_{\text{irr}} = \frac{R_\star}{R} + \left(\frac{H}{R} \right) \left(\frac{d \ln H}{d \ln R} - 1 \right)$$

$$\sigma_{\text{SB}} T_s^4 = \frac{L_\star}{4\pi R^2} \sin \theta_{\text{irr}} \approx \frac{1}{4} \left(\frac{R_\star}{R} \right)^2 \frac{H}{R} \sigma_{\text{SB}} T_\star^4$$

Much less steep!



Note: T_s depends on H/R , which depends on midplane temperature T_c

DISC THERMAL STRUCTURE

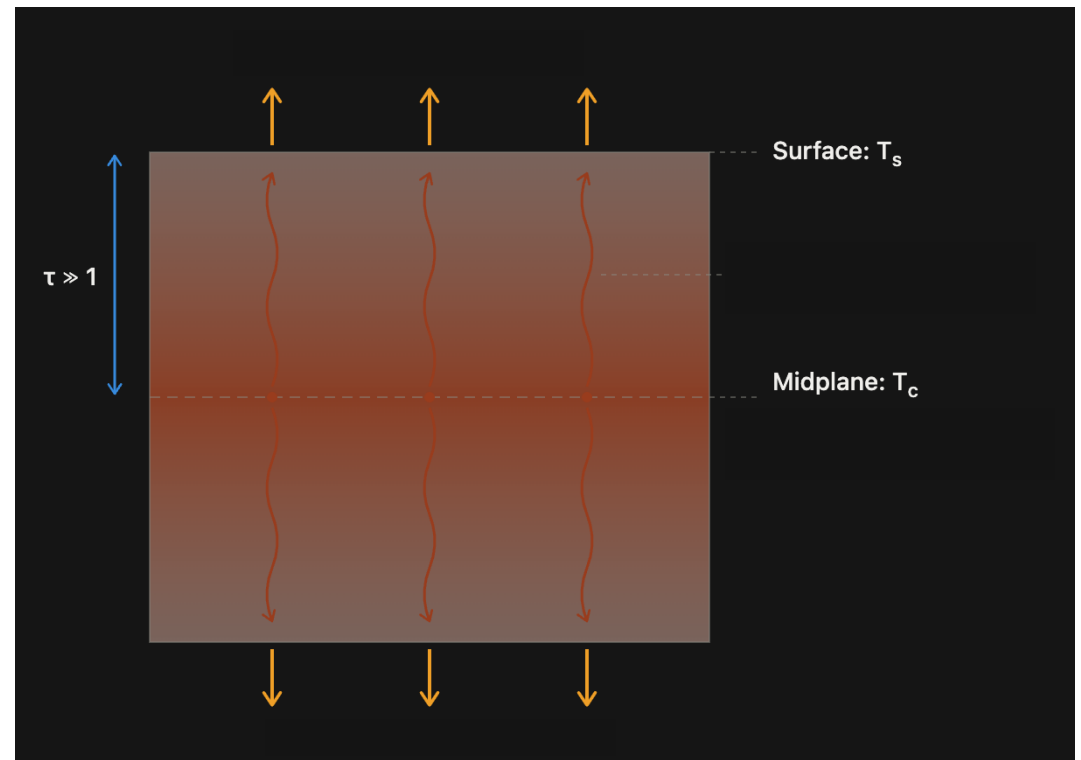
- How to relate surface and midplane temperature

- **Internally heated disc**

$$T_c^4 = \tau T_s^4$$

τ is the optical thickness $\tau = \kappa \Sigma$

Surface is colder than midplane



DISC THERMAL STRUCTURE

- How to relate surface and midplane temperature

- **Irradiated discs**

- Need a self-consistent solution

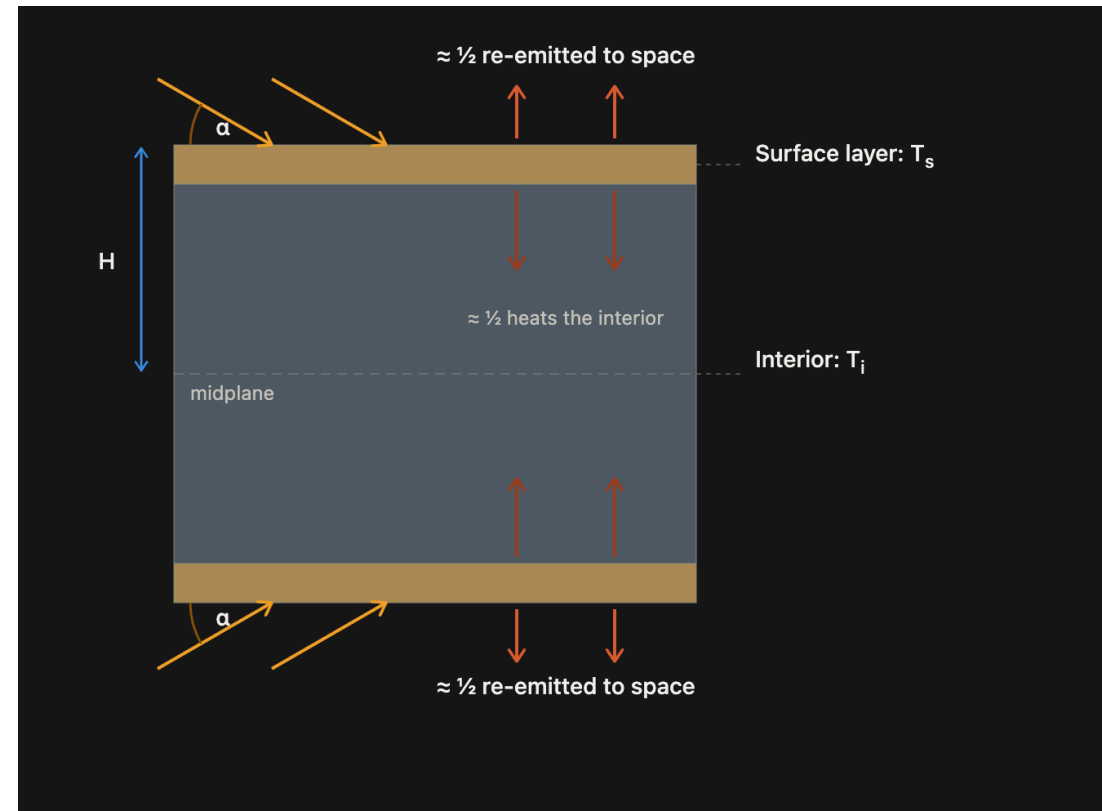
(Chiang & Goldreich 1997)

$$T_c^4 = \frac{\theta_{\text{irr}}}{2} \left(\frac{R_\star}{R} \right)^2 T_\star^4$$

Self-consistent solution in the inner disc has

$$T_c \propto T_s \propto R^{-3/7}$$

In this case, $T_c < T_s$



MULTI-COLOUR BLACKBODY SPECTRUM

- Given a temperature profile $T_s(R)$, we can easily calculate the disc emission as a sum of blackbody spectra with different temperatures

$$\nu F_\nu = \lambda F_\lambda = \frac{\cos i}{D^2} \int_{R_{\text{in}}}^{R_{\text{out}}} \frac{2h\nu^4}{c^2} \frac{2\pi R dR}{e^{h\nu/kT_s(R)} - 1}$$

MULTI-COLOUR BLACKBODY SPECTRUM

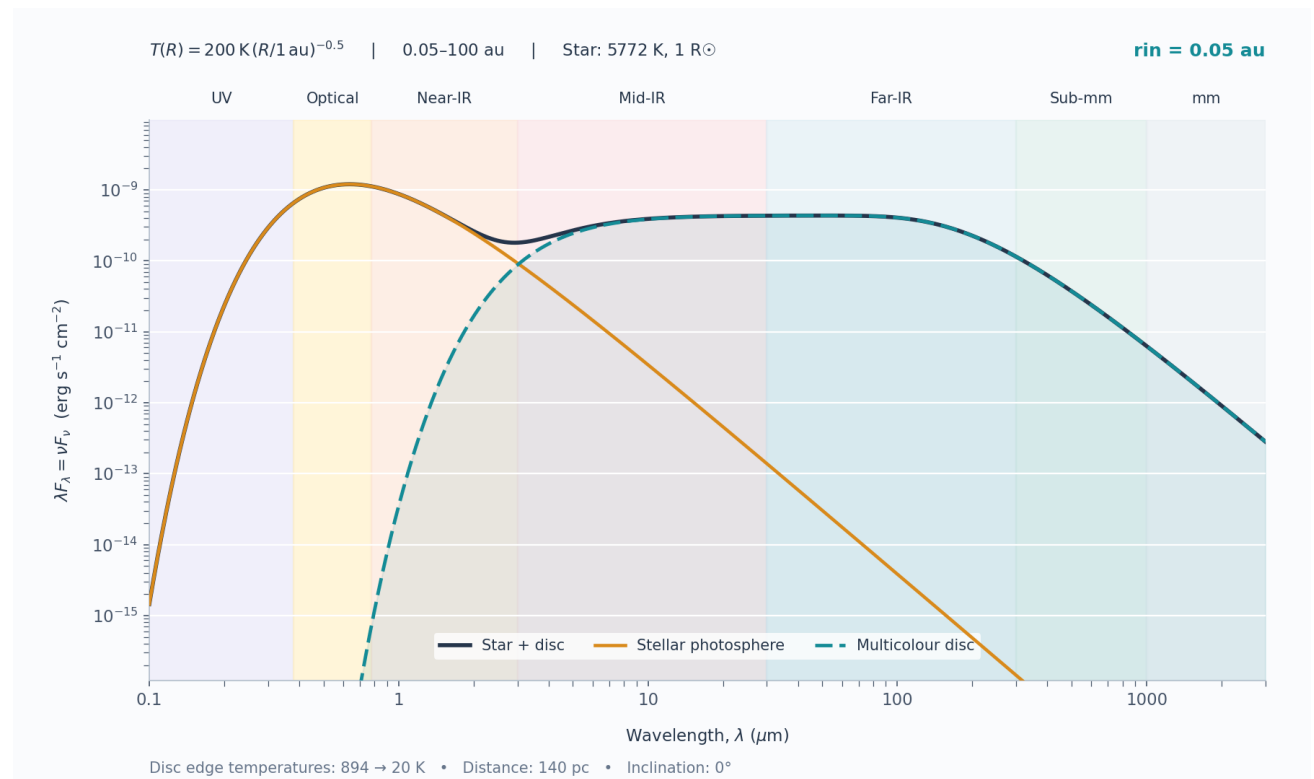
- Given a temperature profile $T_s(R)$, we can easily calculate the disc emission as a sum of blackbody spectra with different temperatures

$$\nu F_\nu = \lambda F_\lambda = \frac{\cos i}{D^2} \int_{R_{\text{in}}}^{R_{\text{out}}} \frac{2h\nu^4}{c^2} \frac{2\pi R dR}{e^{h\nu/kT_s(R)} - 1}$$

Exercise 4: show that, if $T_s(R) \propto R^{-q}$, then $\lambda F_\lambda \propto \lambda^{2/q-4}$ over a wide wavelength range

MULTI-COLOUR BLACKBODY SPECTRUM

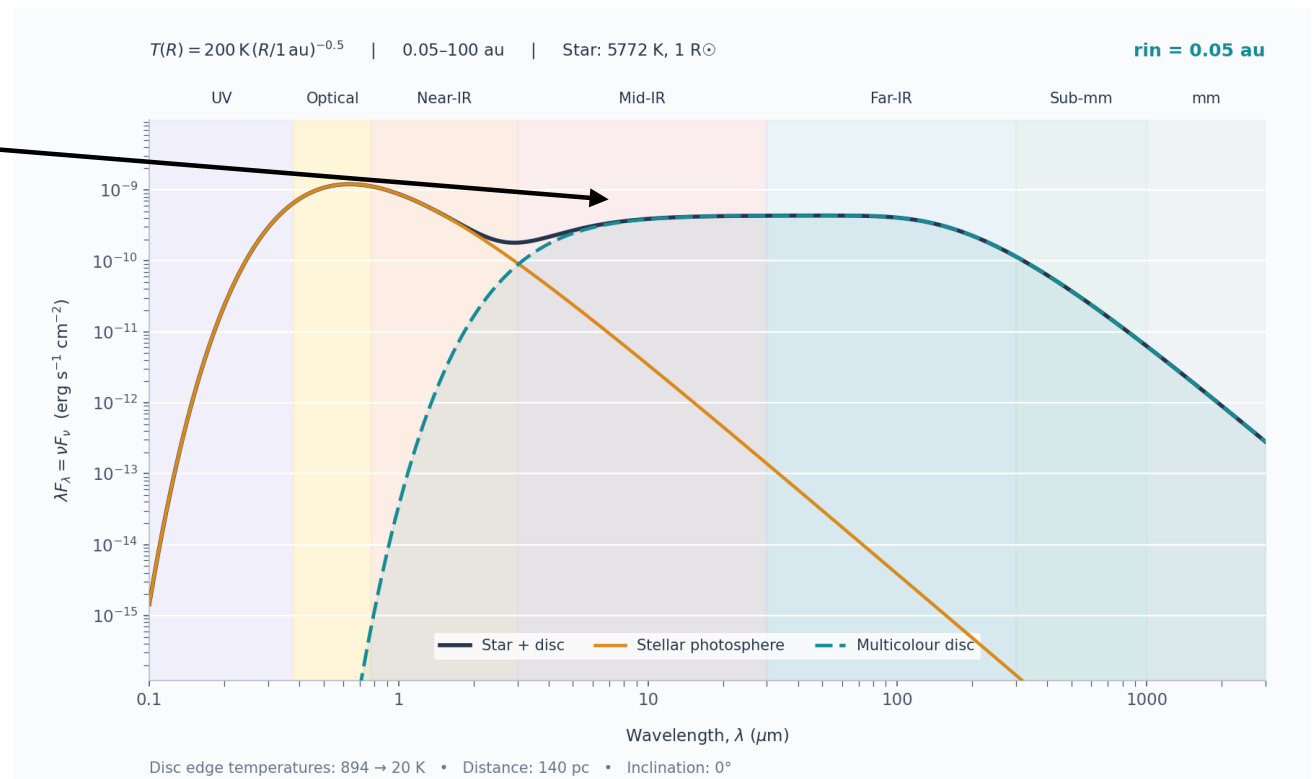
- Given a temperature profile $T_s(R)$, we can easily calculate the disc emission as a sum of blackbody spectra with different temperatures



MULTI-COLOUR BLACKBODY SPECTRUM

- Given a temperature profile $T_s(R)$, we can easily calculate the disc emission as a sum of blackbody spectra with different temperatures

Stars with discs often identified by their “IR excess”

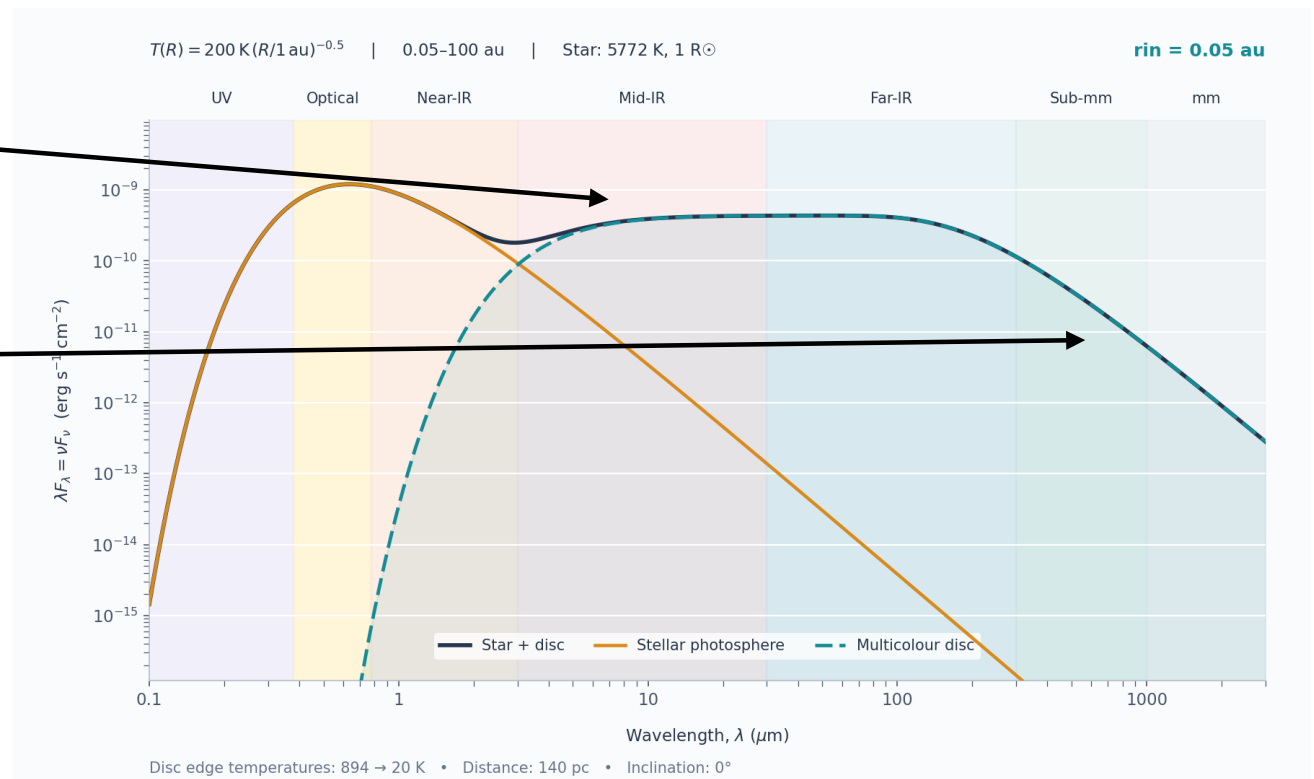


MULTI-COLOUR BLACKBODY SPECTRUM

- Given a temperature profile $T_s(R)$, we can easily calculate the disc emission as a sum of blackbody spectra with different temperatures

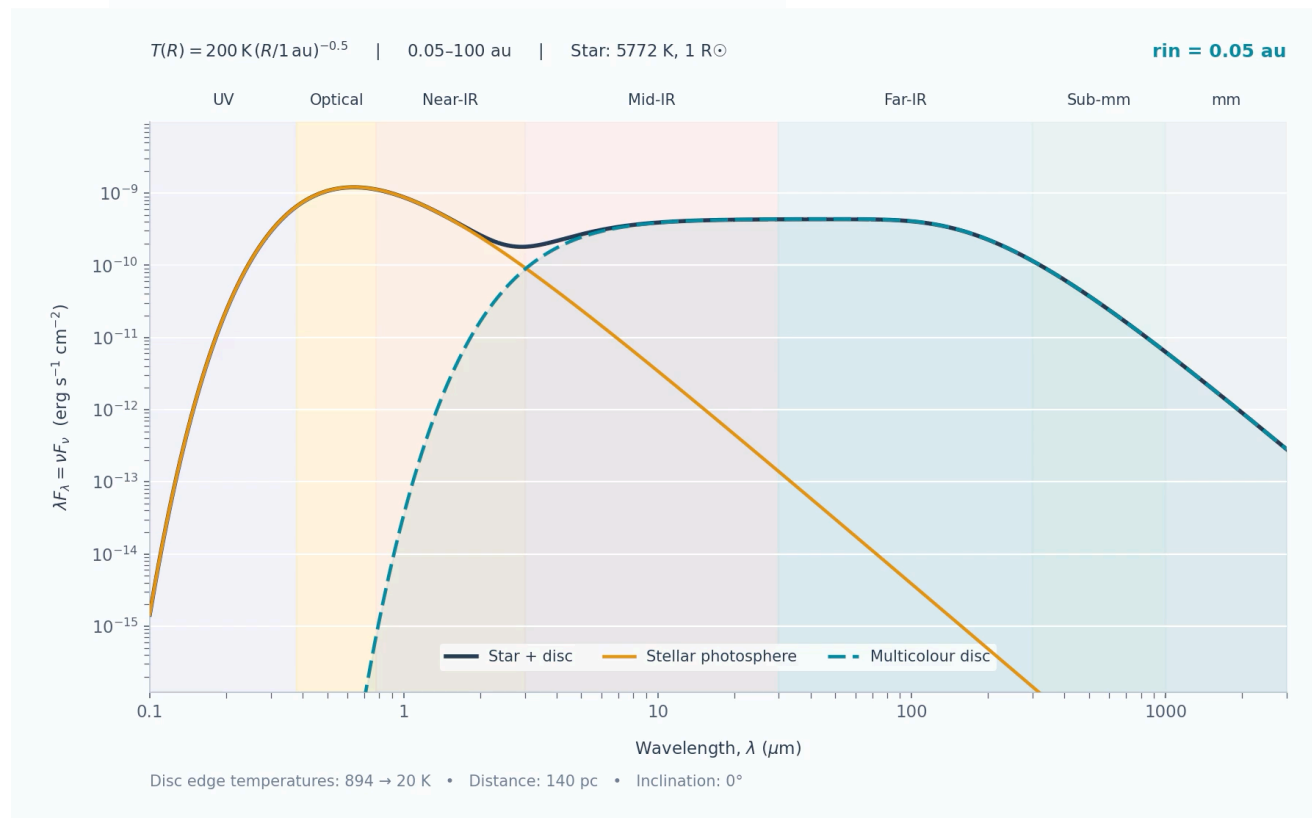
Stars with discs often identified by their “IR excess”

Rayleigh-Jeans tail



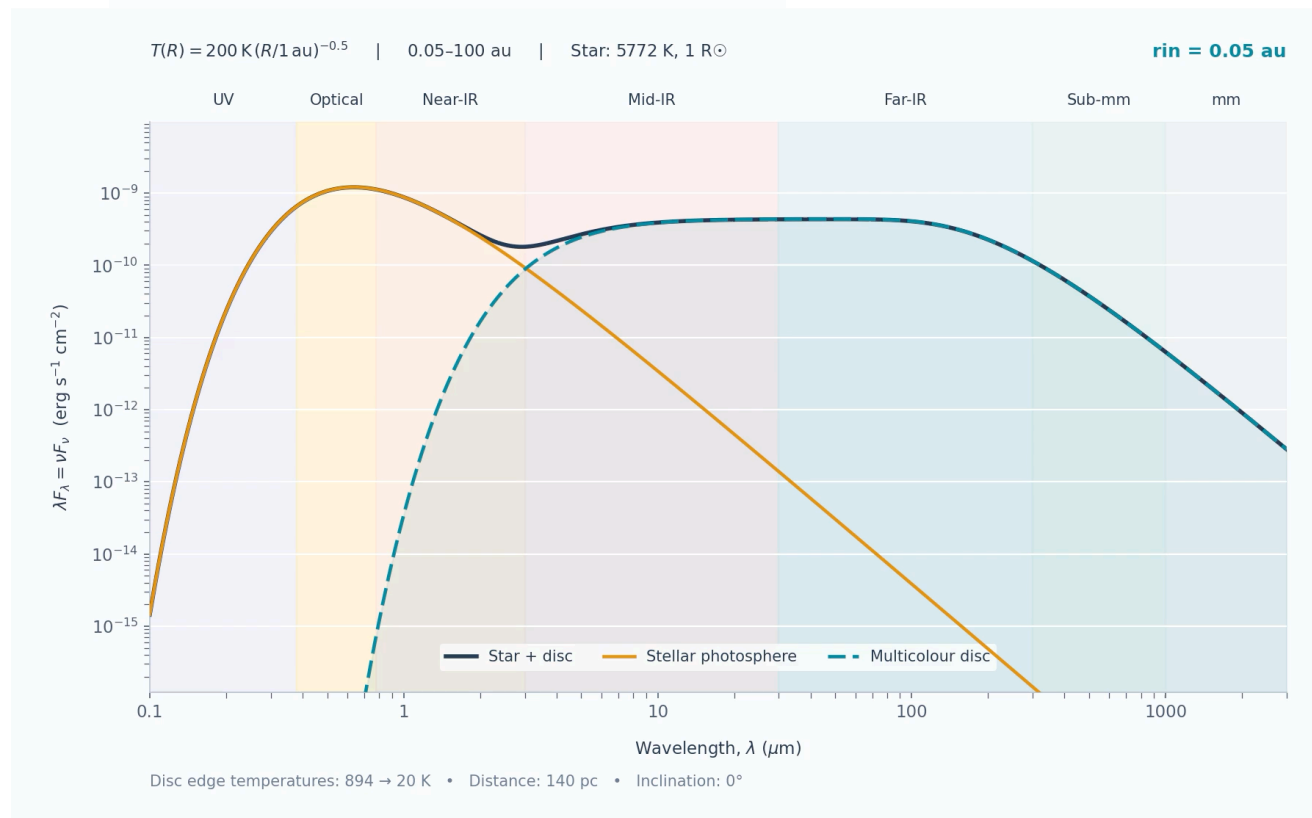
MULTI-COLOUR BLACKBODY SPECTRUM

- Effect of changing the disc inner radius
- Systems with no NIR excess, but FIR excess only are called **transition discs**



MULTI-COLOUR BLACKBODY SPECTRUM

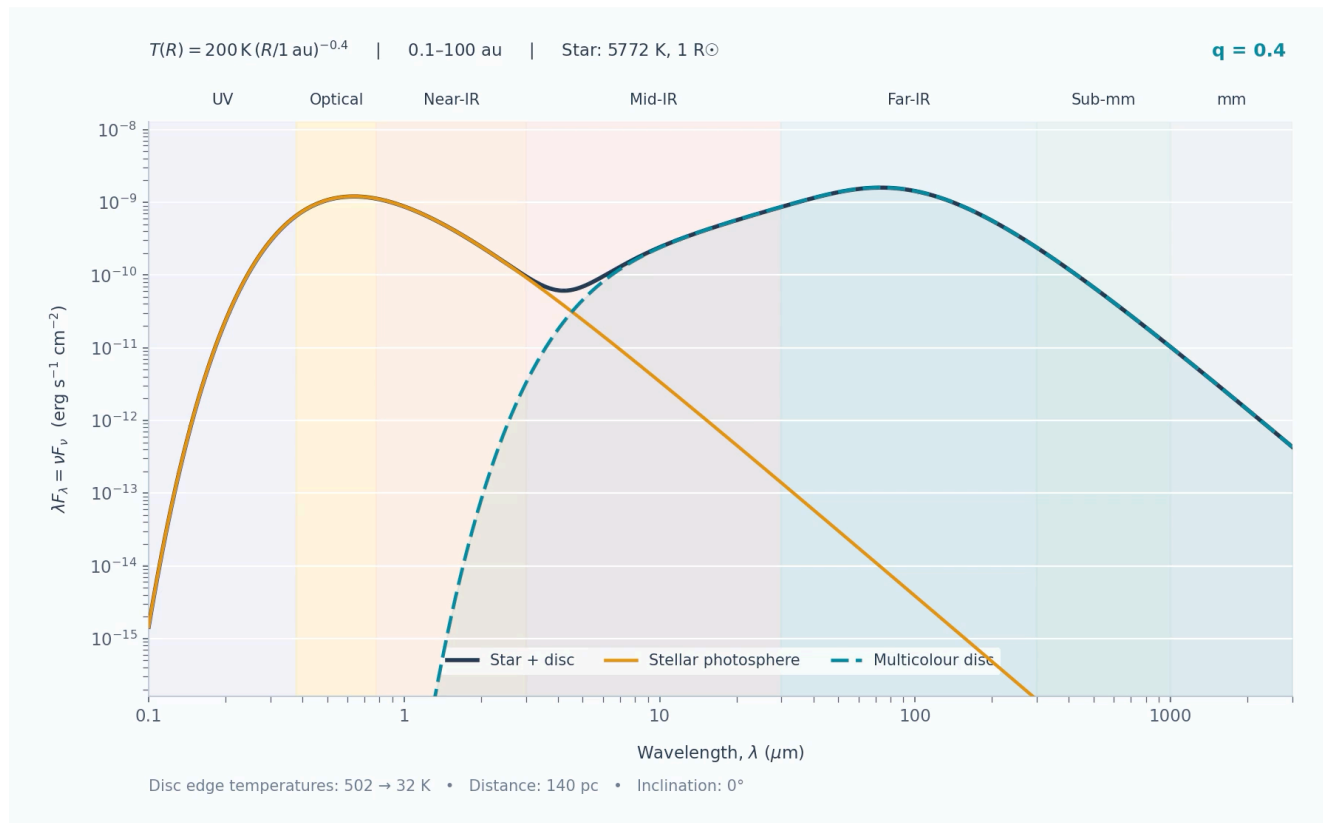
- Effect of changing the disc inner radius
- Systems with no NIR excess, but FIR excess only are called **transition discs**



MULTI-COLOUR BLACKBODY SPECTRUM

- Effect of changing the temperature slope

Remember that $\lambda F_\lambda \propto \lambda^{2/q-4}$



MULTI-COLOUR BLACKBODY SPECTRUM

- Effect of changing the temperature slope

Remember that $\lambda F_\lambda \propto \lambda^{2/q-4}$

