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Italiadomani
PIANO NAZIONALE DI RIPRESA E RESILIENZA

INAF
ISTITUTO NAZIONALE DI ASTROFISICA

Orbit determination techniques for Resident Space Object catalog maintenance

Training Meeting

NG-Croce

Lunedì 12 Maggio - Giovedì 15 Maggio

Radiotelescopi di Medicina

IRA - Bologna

Mauro Massari

Outline

- RSO Catalogs
- Orbit determination problem
- Typical measurements
- Minimum set of measurements
- Initial Orbit Determination
    - Gauss' method
    - Effect of uncertainties
- Precise Orbit Determination
    - Least-square problem
    - Kalman filter

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Resident Space Objects Catalogs

A catalog of resident space objects (RSOs) typically includes detailed information about artificial objects orbiting Earth.

Common elements found in such a catalog:

- **Object Name**: Common name or designation (e.g., Hubble Space Telescope, COSMOS 2251).
- **International Designator**: A standardized code that includes the launch year, launch number, and piece identifier (e.g., 1998-067A).
- **NORAD Catalog Number**: Unique number assigned by the North American Aerospace Defense Command.
- **Object Type**: Classification such as satellite, rocket body, debris, or payload.
- **Owner/Operator**: Country or organization responsible for the object.
- **Launch Date**: Date the object was placed into orbit.
- **Launch Vehicle**: The rocket used to deploy the object.
- **Orbit Type**: General classification (e.g., LEO – Low Earth Orbit, MEO – Medium Earth Orbit, GEO – Geostationary Orbit, HEO – Highly Elliptical Orbit).
- **Current Status**: Active, inactive, decayed, or unknown.
- **Radar Cross Section** (RCS): Estimate of size or detectability by radar.
- **Decay Date**: If applicable, the date the object re-entered Earth's atmosphere.
- **Mission or Function**: Intended use or mission purpose (e.g., communications, Earth observation, scientific research).
- **Tracking Information**: Most recent TLE (Two-Line Element set) or ephemeris data for object tracking.

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Resident Space Objects Catalogs

A catalog of resident space objects (RSOs) serves multiple critical functions in space operations, safety, and policy.

- **Collision Avoidance**: Enables satellite operators to predict and prevent potential collisions (conjunctions) between satellites or with debris.
- **Orbital Traffic Management**: Assists space agencies and private companies in planning satellite launches and maneuvers to avoid overcrowding or unsafe orbital paths.
- **Tracking Space Debris**: Identifies and monitors debris objects to assess risks and develop mitigation strategies.
- **Re-entry Prediction**: Supports the forecasting of when and where objects will re-enter Earth's atmosphere, important for safety and risk assessment.
- **Anomaly Investigation**: Helps investigate unexpected satellite behavior, such as loss of contact or damage, by analyzing nearby RSOs.
- **Research and Analysis**: Provides data for scientific studies on orbital mechanics, long-term space sustainability, and debris population growth.
- **Historical Recordkeeping**: Maintains a database of past and present objects for archival, forensic, and legal purposes.

Resident Space Objects Catalogs Maintenance

RSO Catalogs Are Updated:

1. **Detection**: Sensors detect new objects or movements (e.g., from launches, breakups, or maneuvers).
2. **Identification**: Analysts determine whether it's a new object, a known one with updated data, or a fragmentation/debris piece.
3. **Orbit Determination**: Orbital elements are calculated, often using the **TLE (Two-Line Element)** format for public distribution.
4. **Tracking**: Multiple observations are used to refine the orbit using radar, optical telescopes, and infrared sensors.
5. **Catalog Entry or Update**: The catalog is updated with the new object's details, or the trajectory of an existing object is revised.

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Orbit Determination

Orbit determination problem

Orbit Determination Problem

- Orbit determination consists in computing the state of an orbiting object at a given epoch starting from a set of measurements
  - Compute the orbit of an newly discovered asteroid observed with a telescope
  - Compute the position and velocity of a spacecraft using measurements obtained by Earth-based systems or on-board sensors

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⇩

All missions require spacecraft navigation and orbit determination

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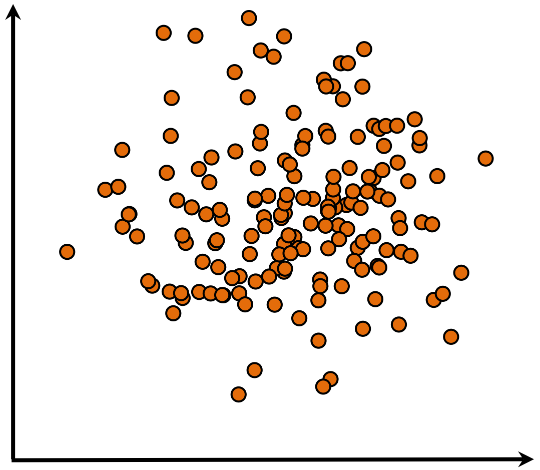
Orbit Determination Problem

- Do we actually know exactly where a spacecraft is?

Orbit Determination Problem

- Do we actually know exactly where a spacecraft is?
- No, there are many sources of error
  - Modeling errors
  - Launch errors
  - Spacecraft performance
  - Observation errors



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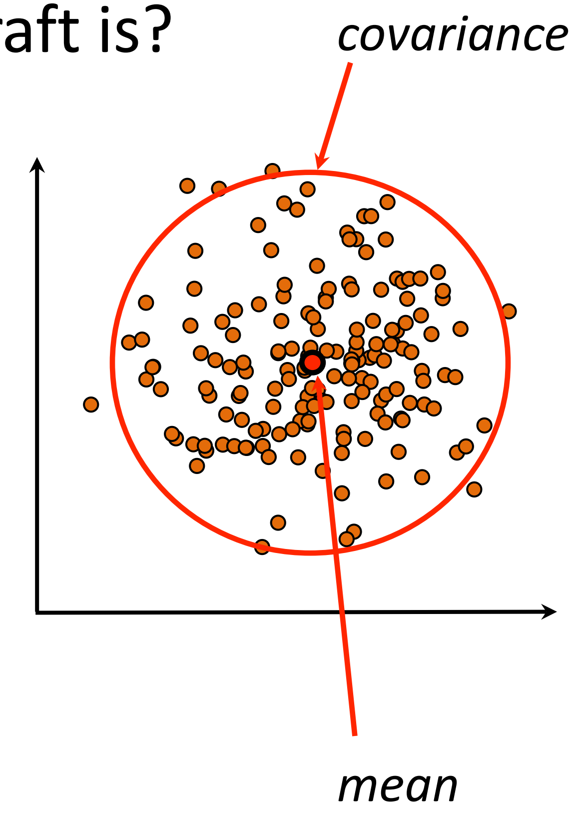
Orbit Determination Problem

- Do we actually know exactly where a spacecraft is?
- No, there are many sources of error
  - Modeling errors
  - Launch errors
  - Spacecraft performance
  - Observation errors
- The set of possible positions of the spacecraft is usually described by the mean state and the covariance matrix C



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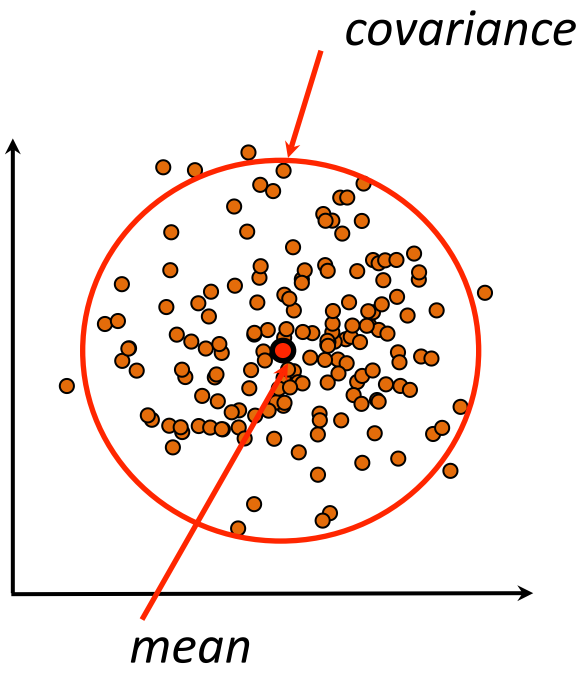
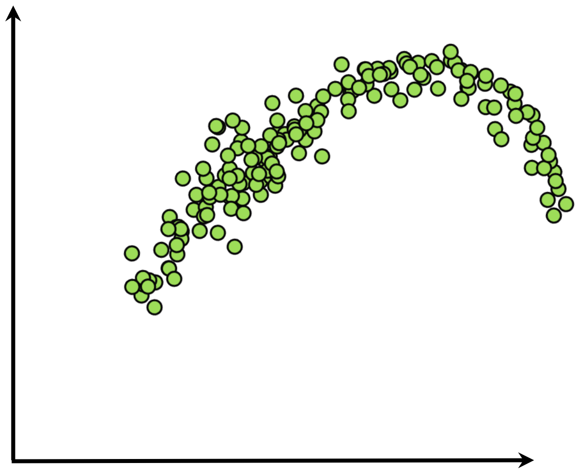
Orbit Determination Problem

- Orbit determination objective: estimating the mean and covariance describing the state of the object
  - Be careful: mean and covariance are not always the best choice





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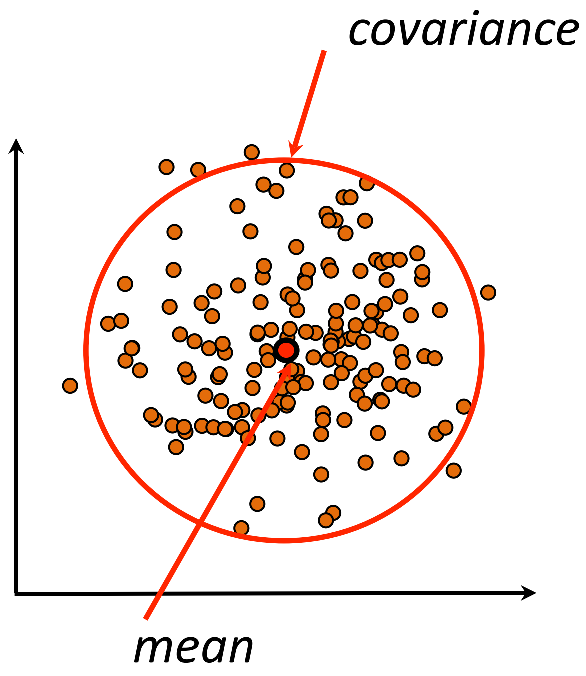

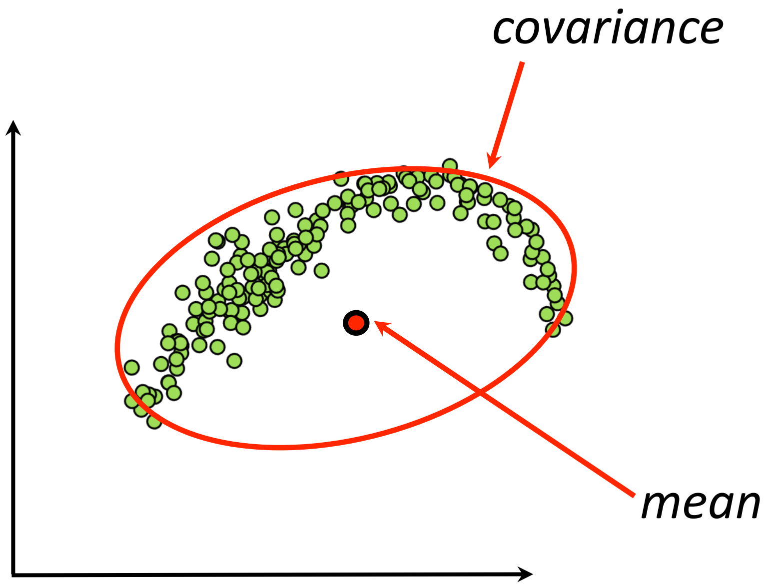

Orbit Determination Problem

- What is the accuracy level we need on the knowledge of the spacecraft state?
- For most situations in spacecraft navigation:
  - Positions: meter level accuracy
  - Velocities: m/s accuracy

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Orbit Determination Problem

- What is the accuracy level we need on the knowledge of the spacecraft state?
- For most situations in spacecraft navigation:
  - Positions: meter level accuracy
  - Velocities: m/s accuracy
- For relative navigation:
  - Positions: cm level accuracy
  - Velocities: cm/s accuracy

Credits: NASA

Orbit Determination

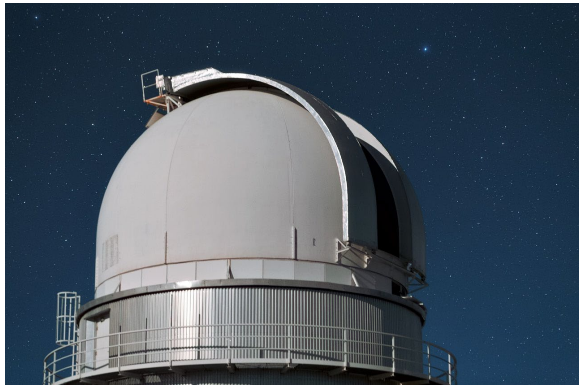
Typical Measurements

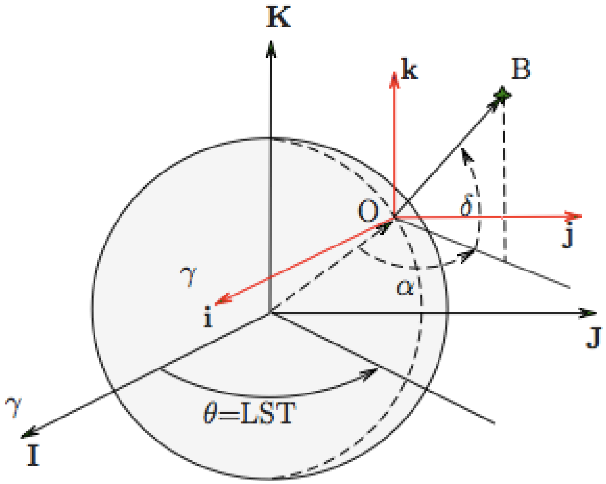
Typical Measurements

- What are the possible measurements to estimate spacecraft position?

Ground measurements

- Optical observations with telescopes:
  - Right ascension α
  - Declination δ





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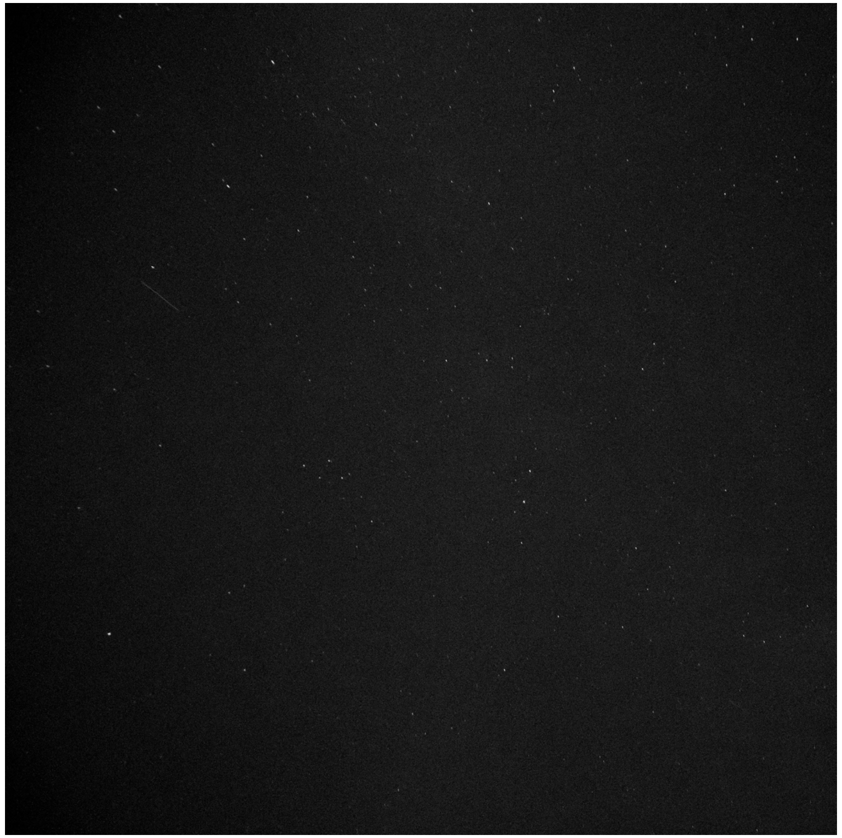
Typical Measurements

- Typical optical observation of an Earth-orbiting spacecraft



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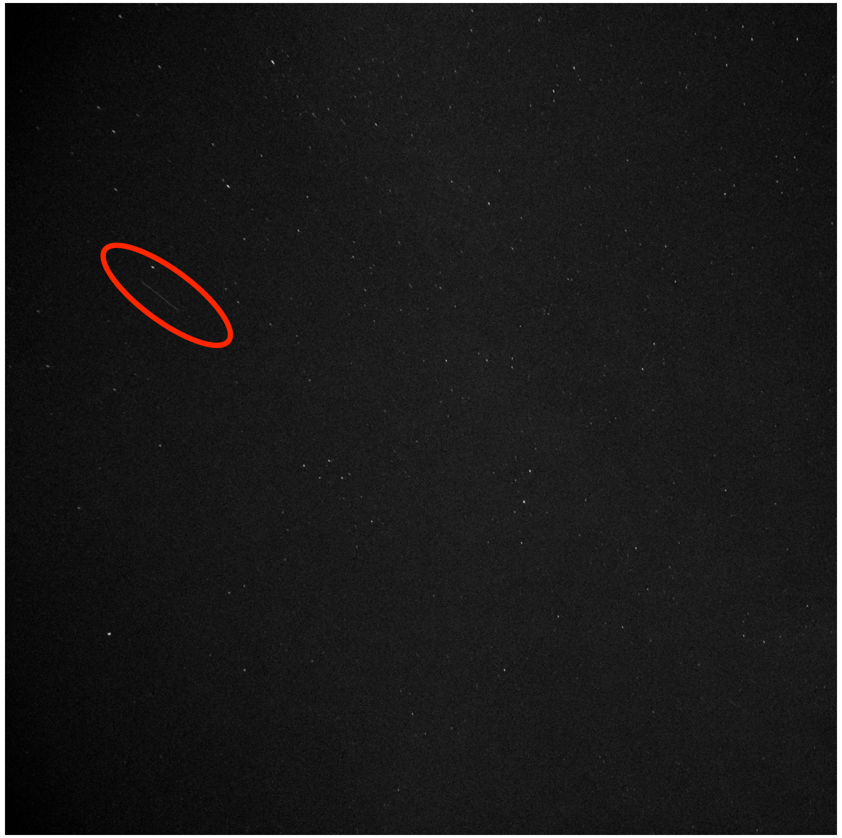

- Identify the tracklet

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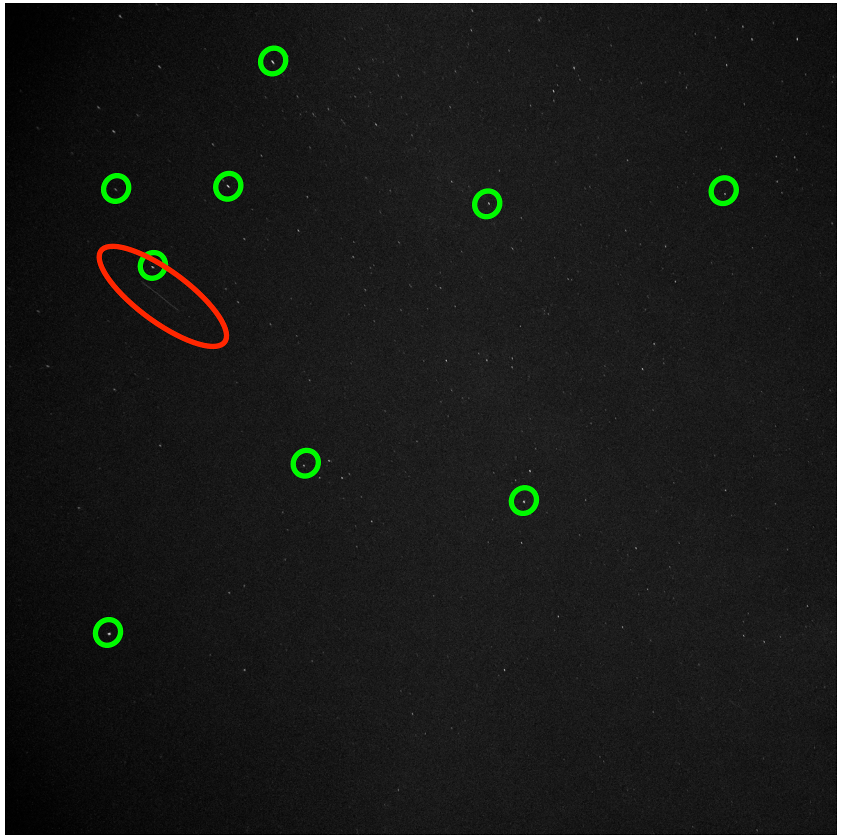

- Identify the tracklet
- Matches observed stars with catalogue

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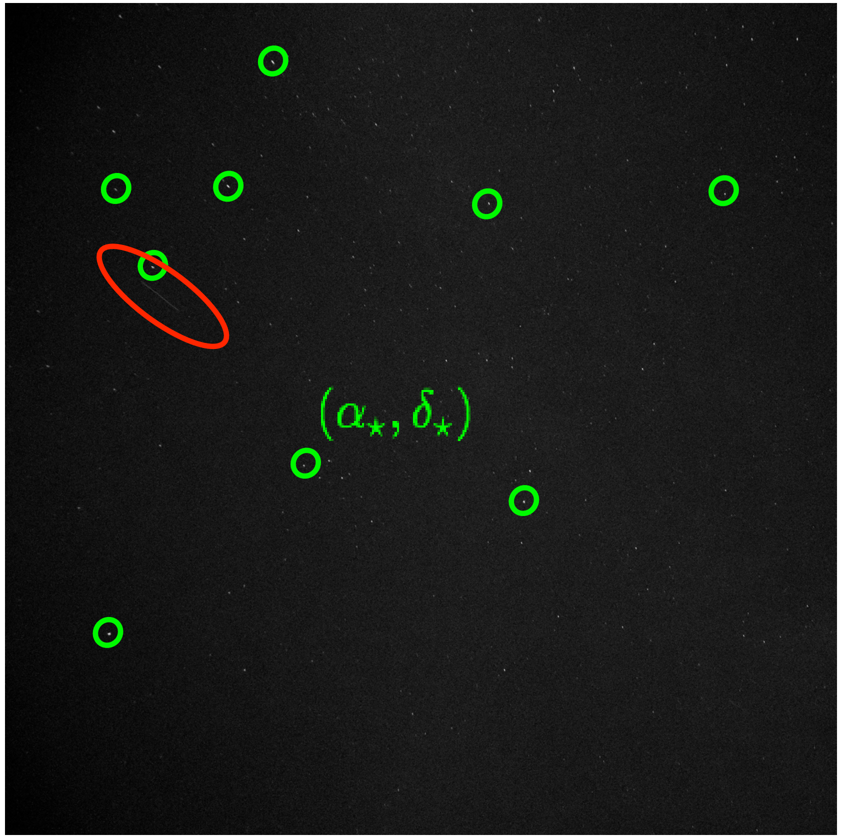
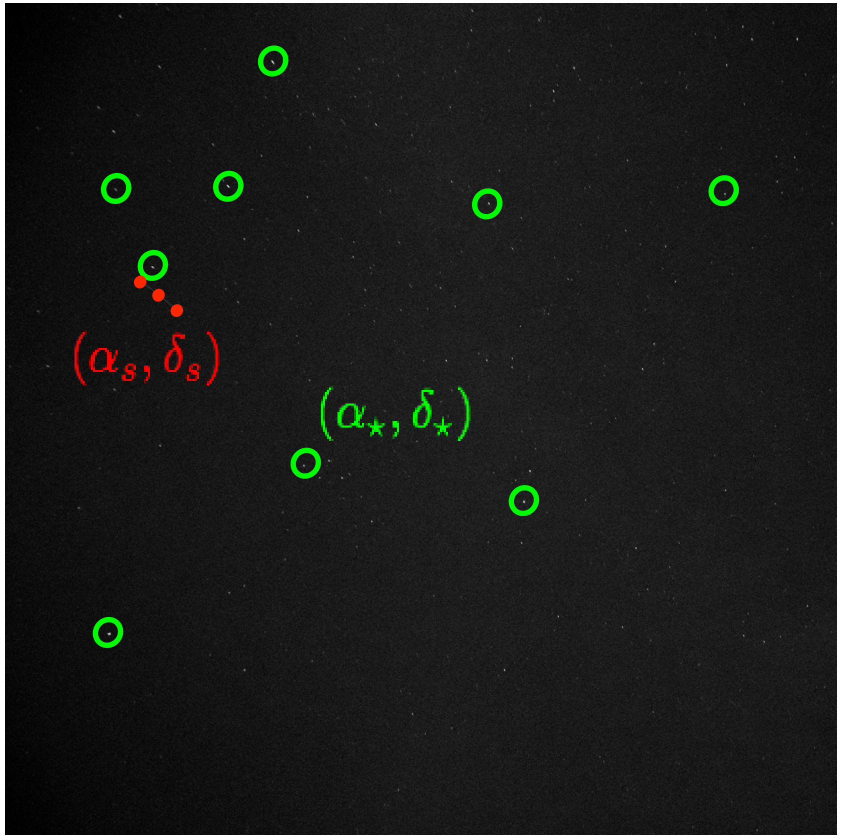


- Identify the tracklet
- Matches observed stars with catalogue
- Retrieve observables of catalogued stars

Typical Measurements

- Typical optical observation of an Earth-orbiting spacecraft



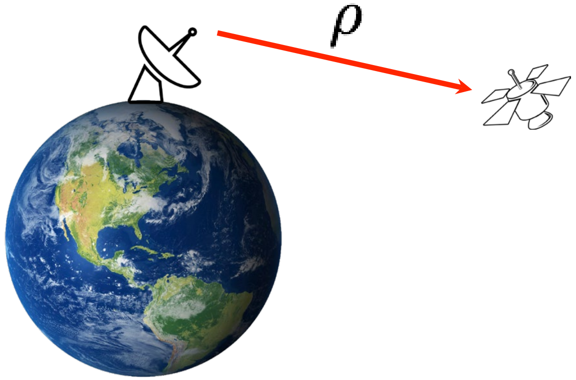
- Identify the tracklet
- Matches observed stars with catalogue
- Retrieve observables of catalogued stars
- Compute observables of the spacecraft w.r.t. catalogued stars

Typical Measurements

- What are the possible measurements to estimate spacecraft position?

Ground measurements

- Radar observations with radar antenna:
  - Range ρ
  - Doppler shift ⇨ range rate $\dot{\rho}$



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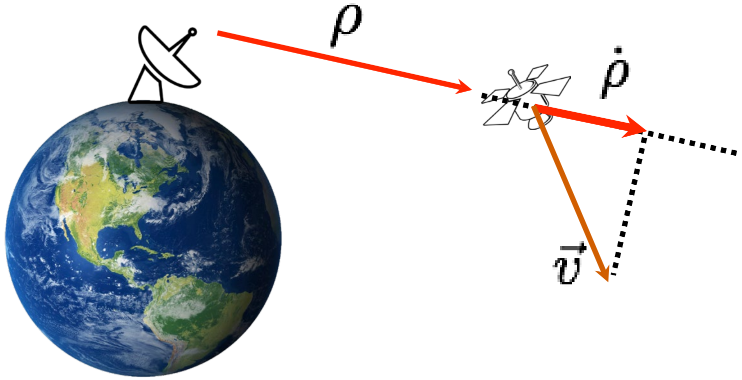

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Typical Measurements

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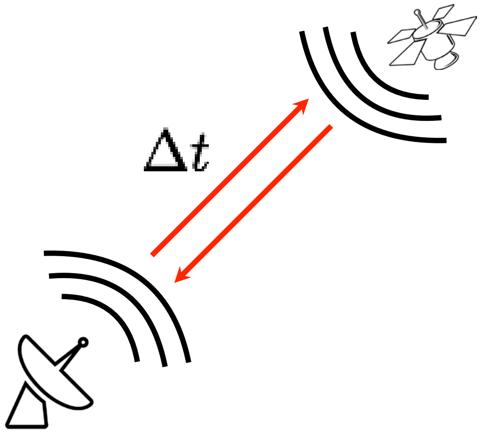
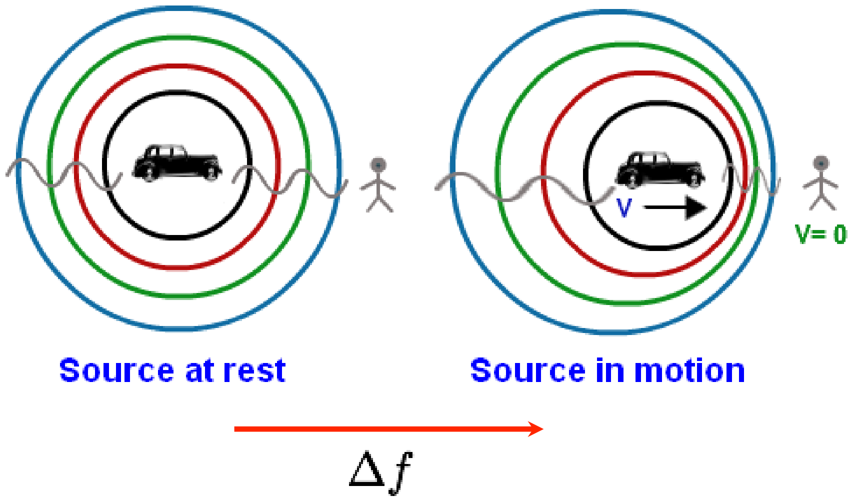


$\Delta t \Rightarrow \rho$

Doppler



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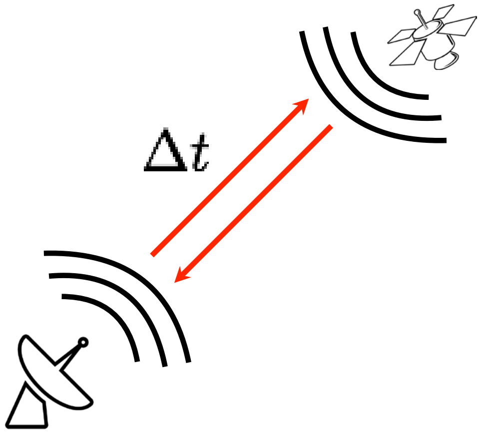

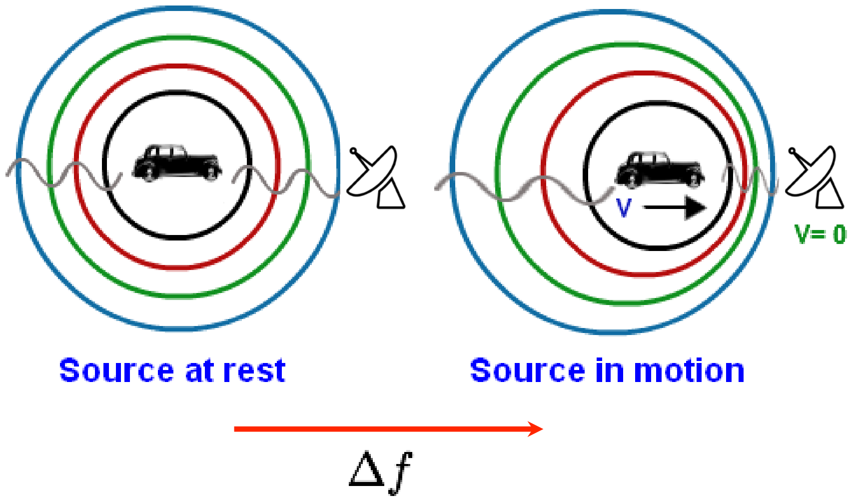

$\Delta t \Rightarrow \rho$

$\Delta f \Rightarrow \dot{\rho}$

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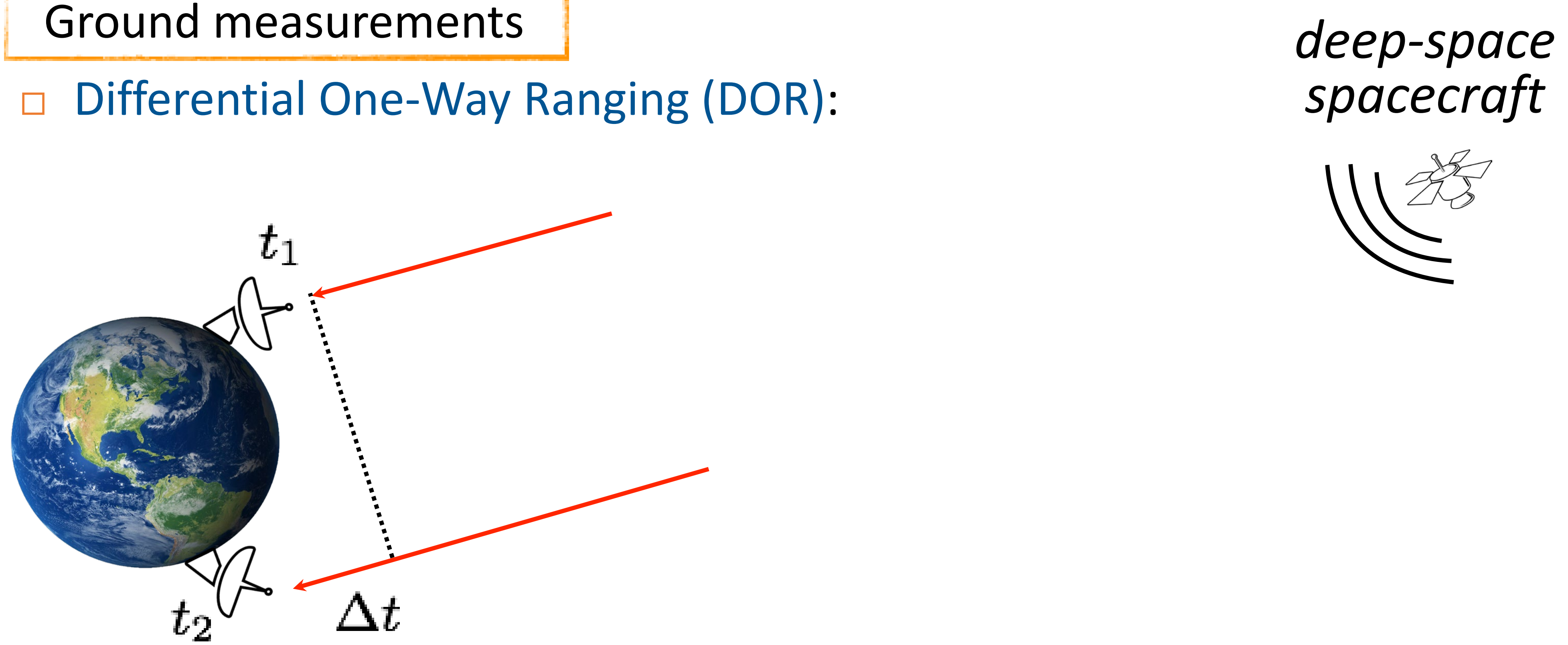
Typical Measurements

- What are the possible measurements to estimate spacecraft position?

Ground measurements

- Differential One-Way Ranging (DOR):



Typical Measurements

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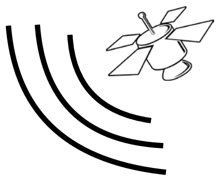
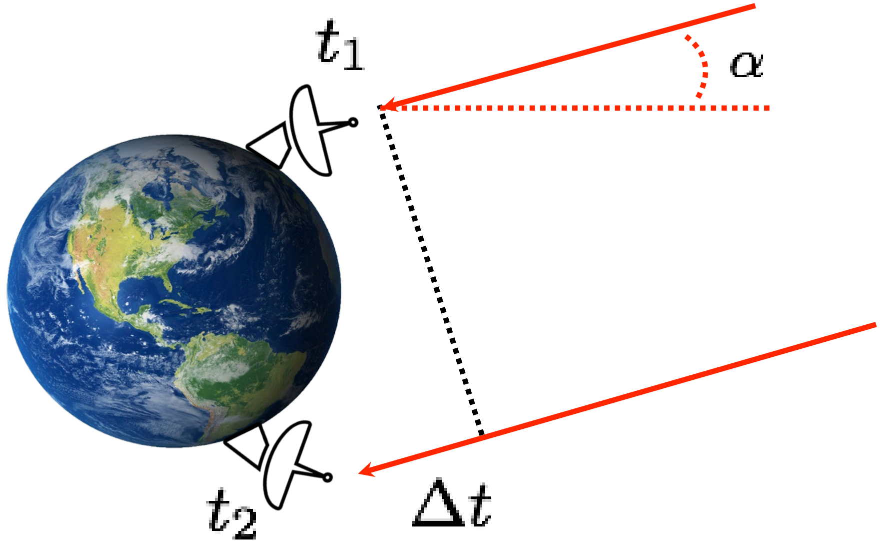
Ground measurements

- Differential One-Way Ranging (DOR):

deep-space spacecraft





$$\boxed{\Delta t \Rightarrow \alpha}$$

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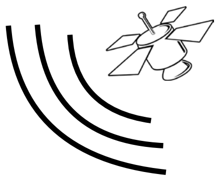

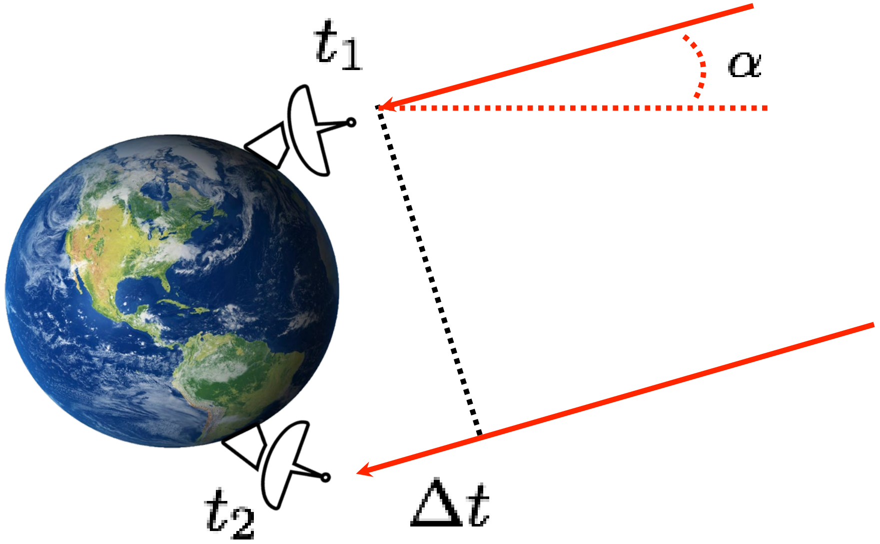

$\boxed{\Delta t \Rightarrow \alpha}$

- Delta-DOR: obtained by correcting DOR with simultaneous tracking of radio signals from a quasar

Typical Measurements

- What are the possible measurements to estimate spacecraft position?

  Onboard measurements

  - Accelerations

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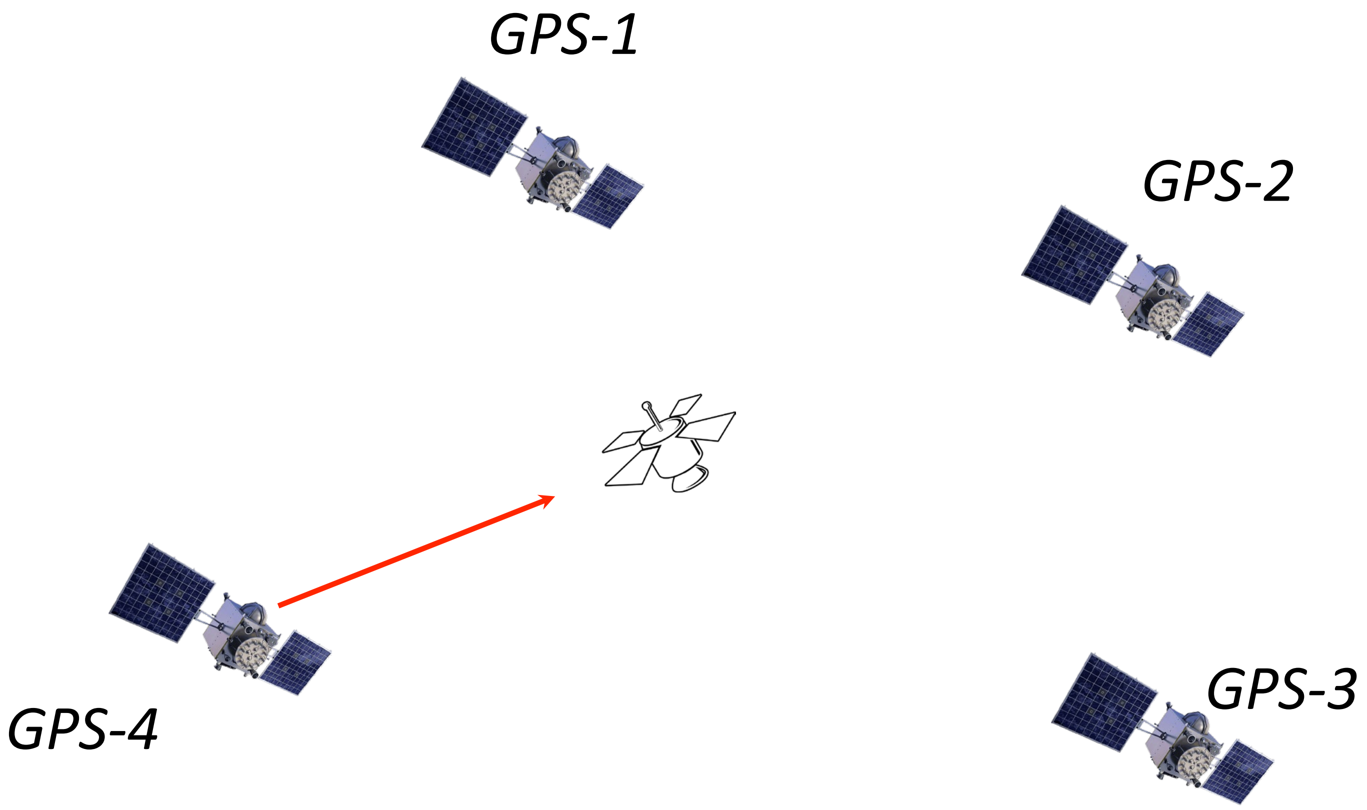
Typical Measurements

- What are the possible measurements to estimate spacecraft position?

Onboard measurements

- Accelerations
- GPS



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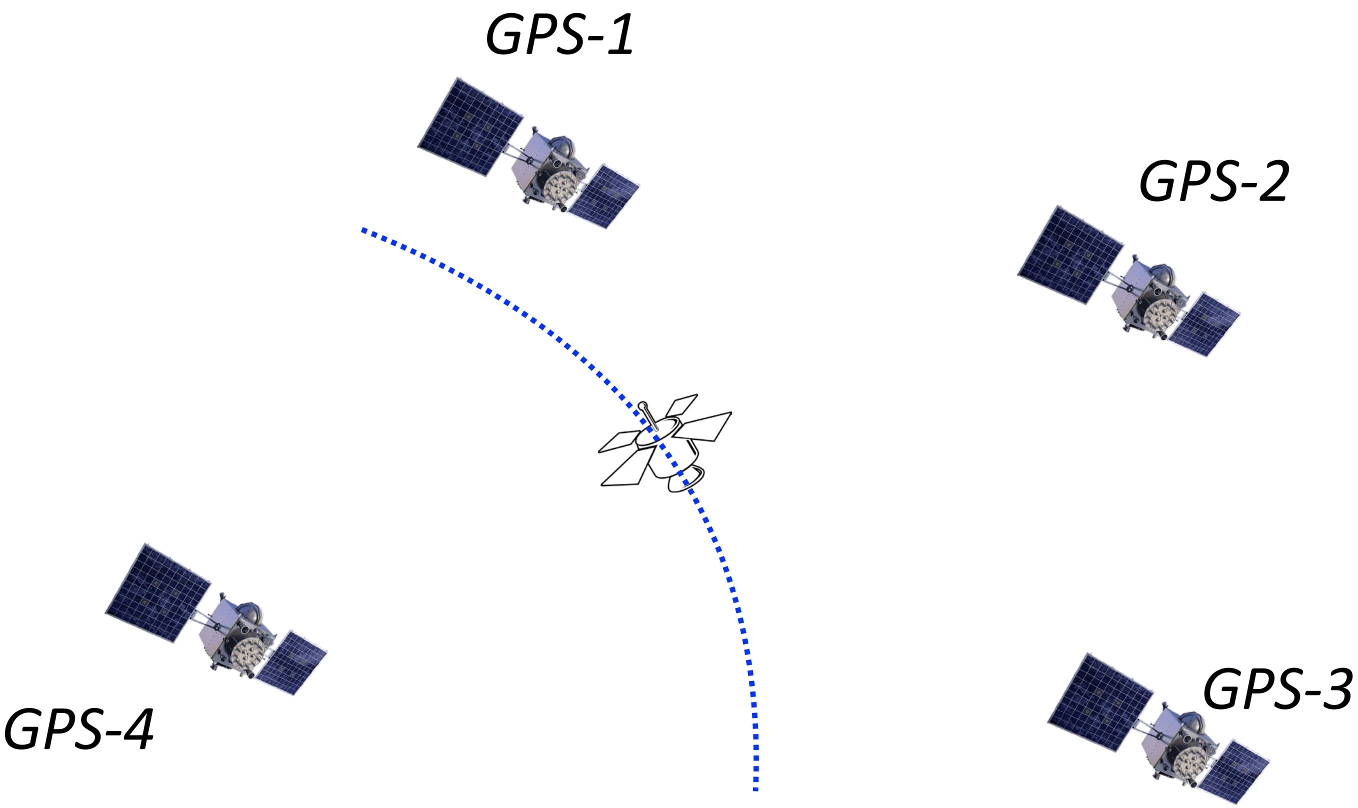

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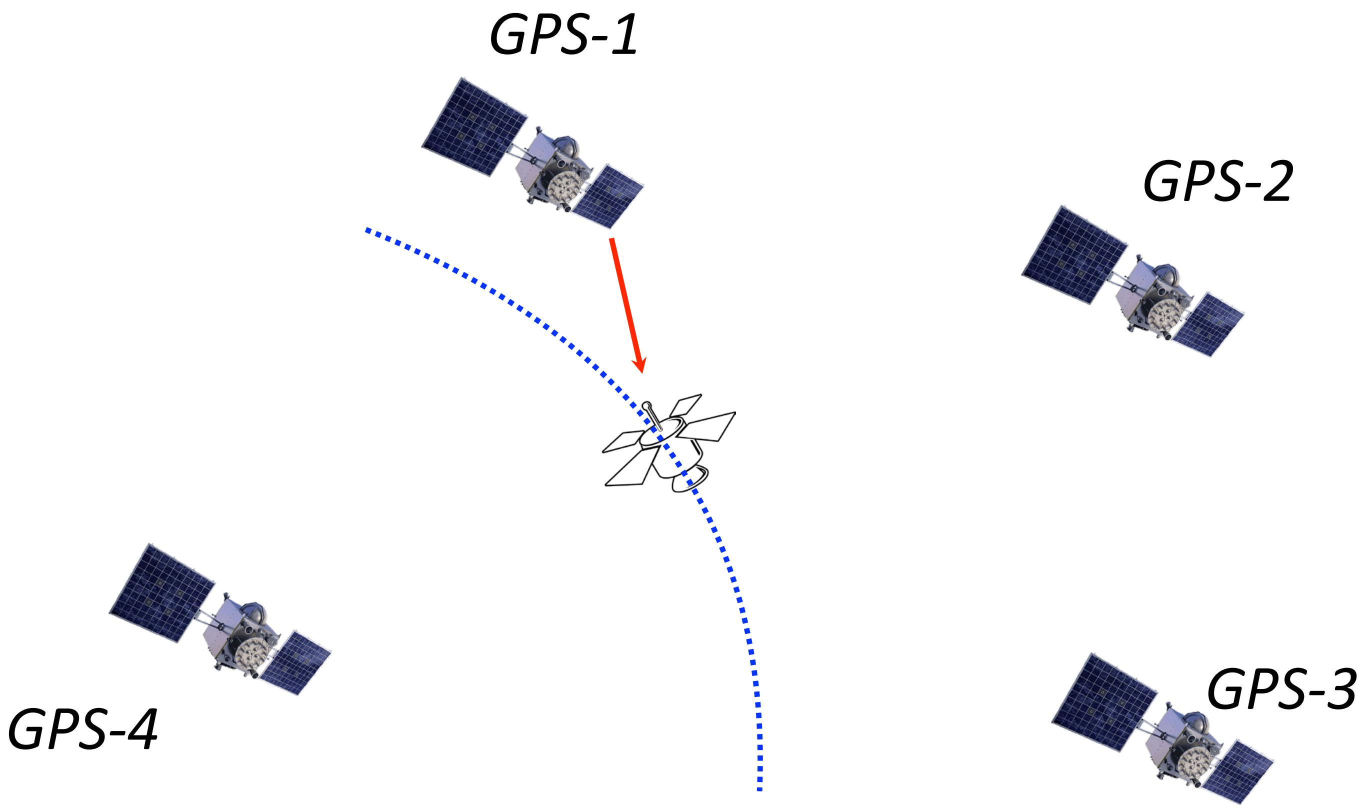

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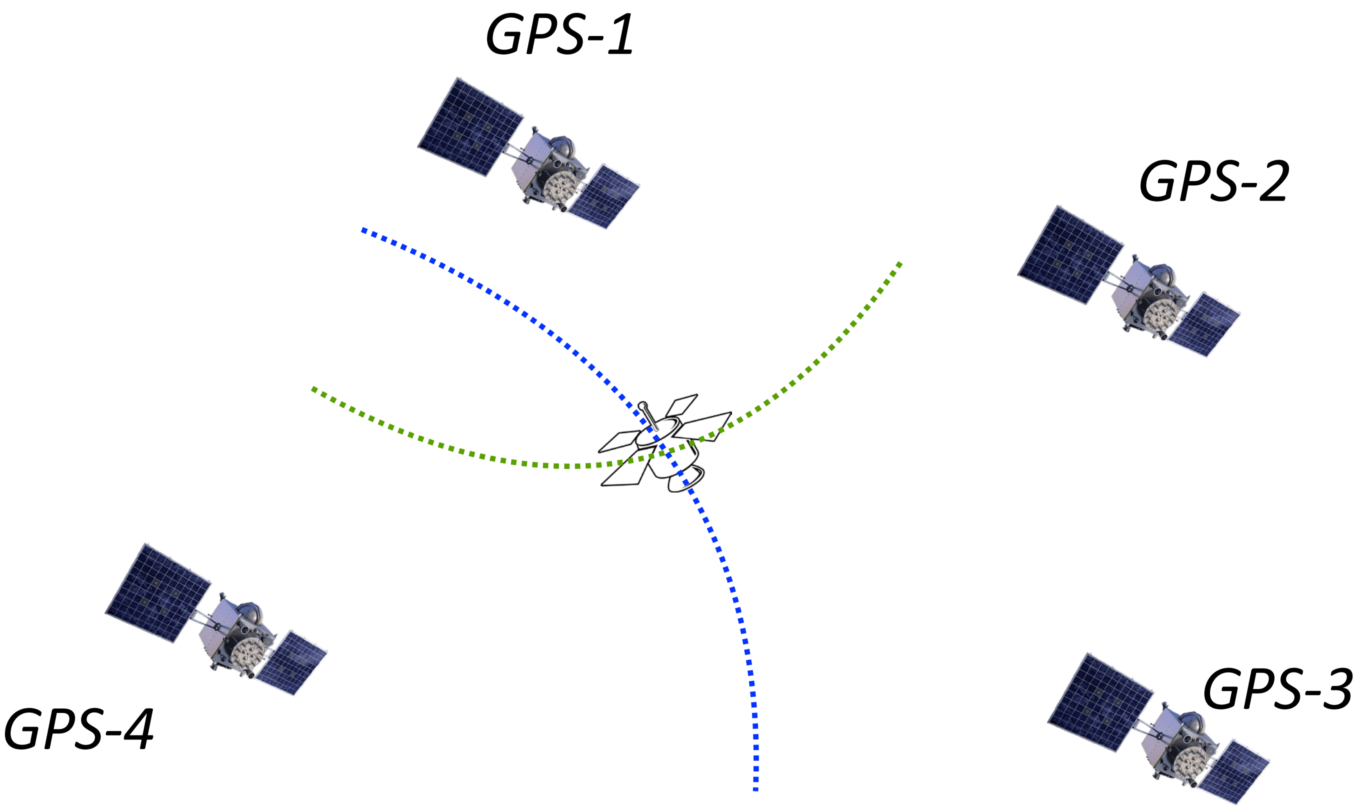

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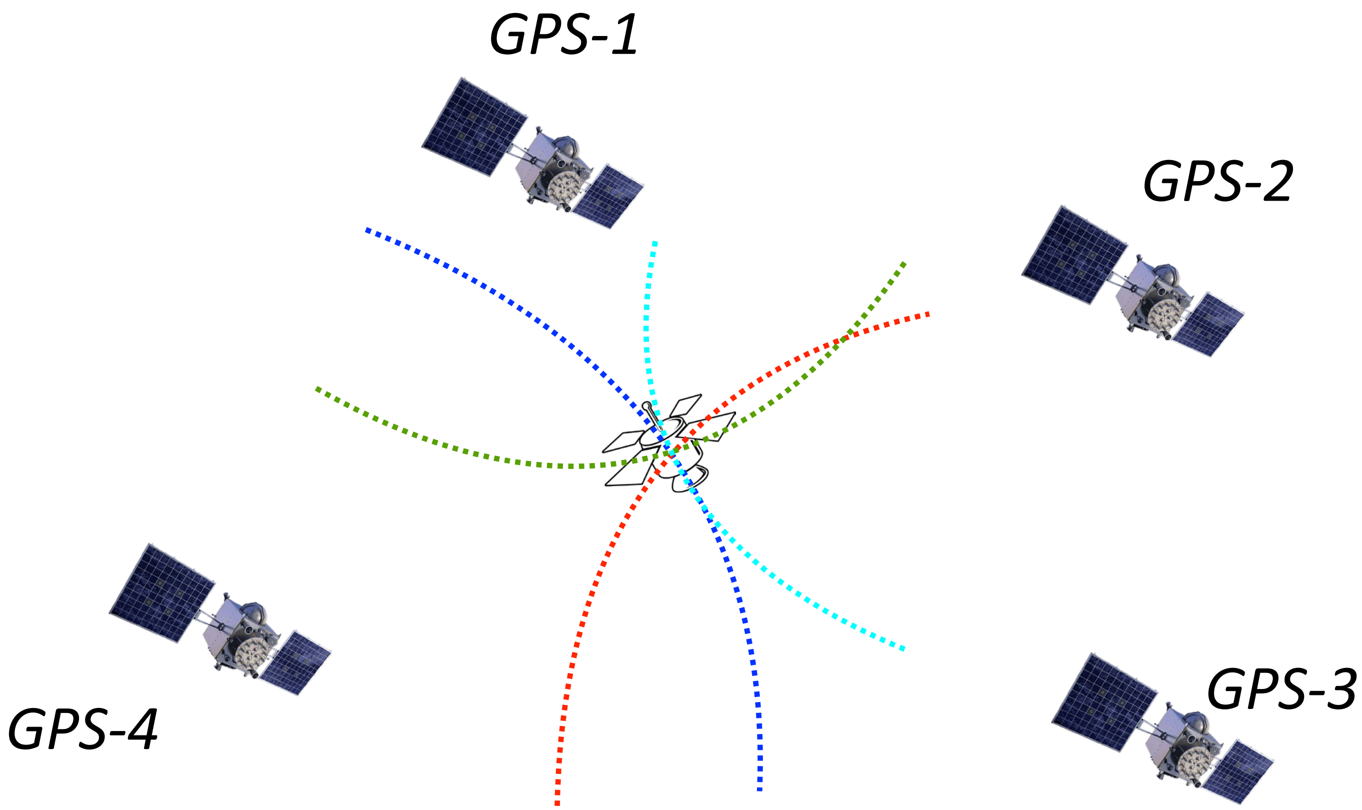


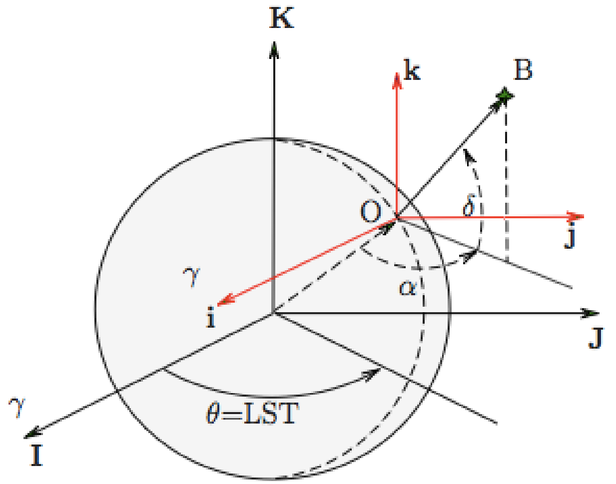
Orbit Determination

Minimum Set of Measurements

Minimum Set of Measurements

- How many measurements do we need to estimate spacecraft state $\vec{x} = \{\vec{r}, \vec{v}\}$?
- Consider one optical observation: α and δ
  - given the position vector of the spacecraft w.r.t. the observer in the inertial reference frame (x, y, z)

$$\begin{cases} \alpha = \operatorname{atan}\dfrac{y}{x} \\ \delta = \operatorname{atan}\dfrac{z}{\sqrt{x^2 + y^2}} \end{cases}$$



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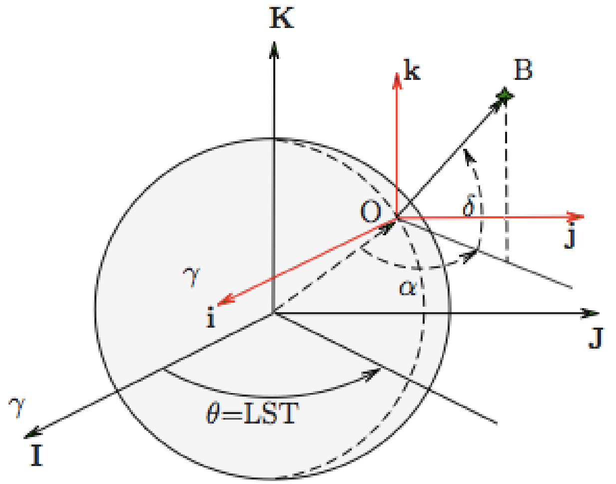

2 equations with 6 unknowns ⇨ *cannot be solved*

Minimum Set of Measurements

- How many measurements do we need to estimate spacecraft state $\vec{x} = \{\vec{r}, \vec{v}\}$?
- Consider one radar observation: ρ and $\dot{\rho}$
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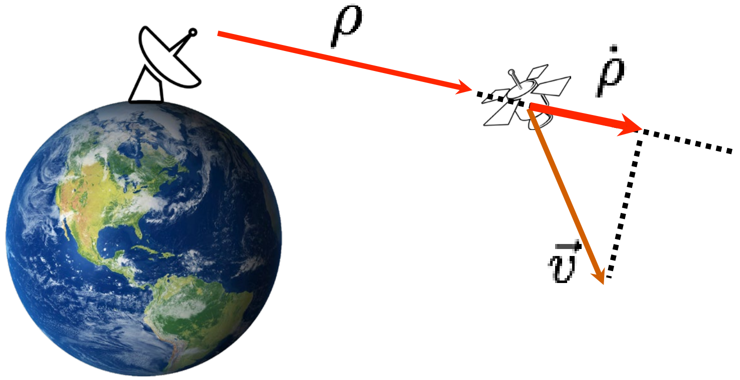
$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \dot{\rho} = \vec{v} \cdot \dfrac{\vec{r}}{\rho} \end{cases}$$



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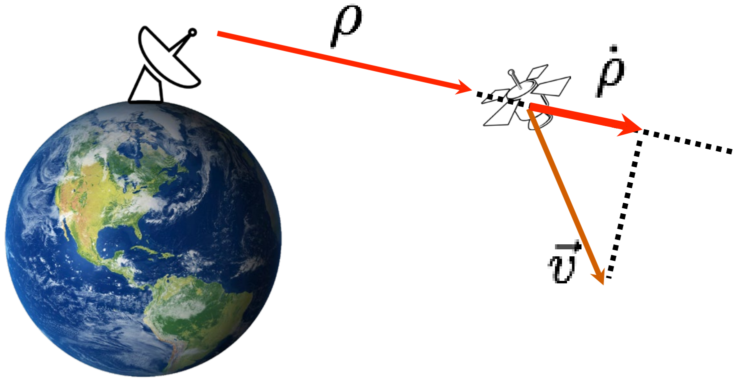

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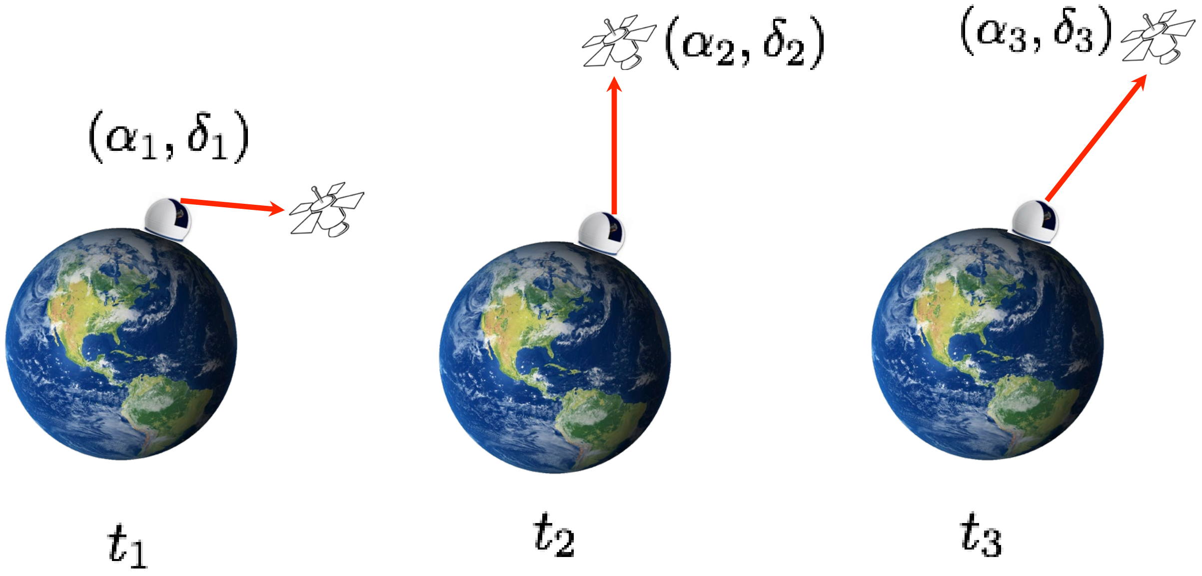

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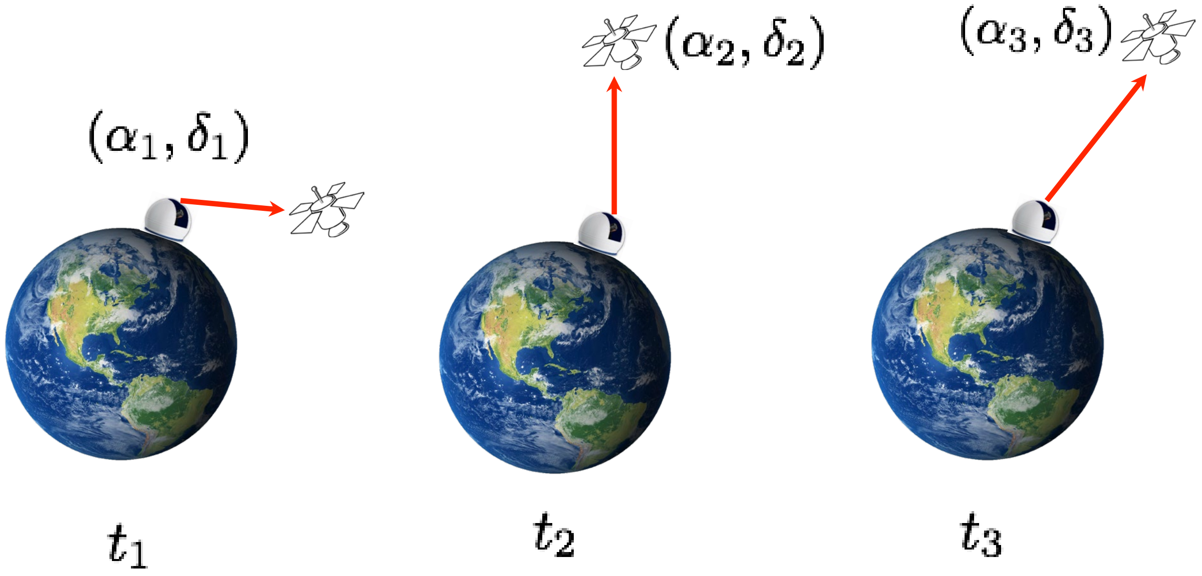

6 equations with 6 unknowns

Measurements

- As the unknowns are 6 (position and velocity vectors) the problem could be:
  1. underdetermined if the number of linearly independent measurements is < 6
  2. well posed if the number of linearly independent measurements is = 6
  3. overdetermined if the number of linearly independent measurements is > 6

Measurements

- As the unknowns are 6 (position and velocity vectors) the problem could be:
  1. underdetermined if the number of linearly independent measurements is < 6
  2. well posed if the number of linearly independent measurements is = 6
  3. overdetermined if the number of linearly independent measurements is > 6

- Associated to each case we have:
  1. infinite solutions, i.e. the problem cannot be solved
  2. one solution (or multiple solutions that can be reduced to one)
  3. the solution is the best solution that better fits all the measurements (least square method)

Orbit Determination Sequence

- Orbit determination is split in two phases
  - **Initial orbit determination**: is the process of determining an initial orbit
    - It is possible when the number of independent measurements is equal to the unknowns (i.e., 6)
    - It requires the solution of a set of nonlinear equations (usually an iterative procedure as Newton's method)
    - It asks for an initial guess or a way to discern from different solutions

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    - It asks for an initial guess or a way to discern from different solutions
  - **Orbit determination refinement**: it is the process of updating the solution exploiting new measurements
    - An initial orbit is necessary
    - The best estimation is obtained by solving a least square problem
    - The covariance on the measurements are mapped in the state vector space

Orbit Determination

Initial Orbit Determination

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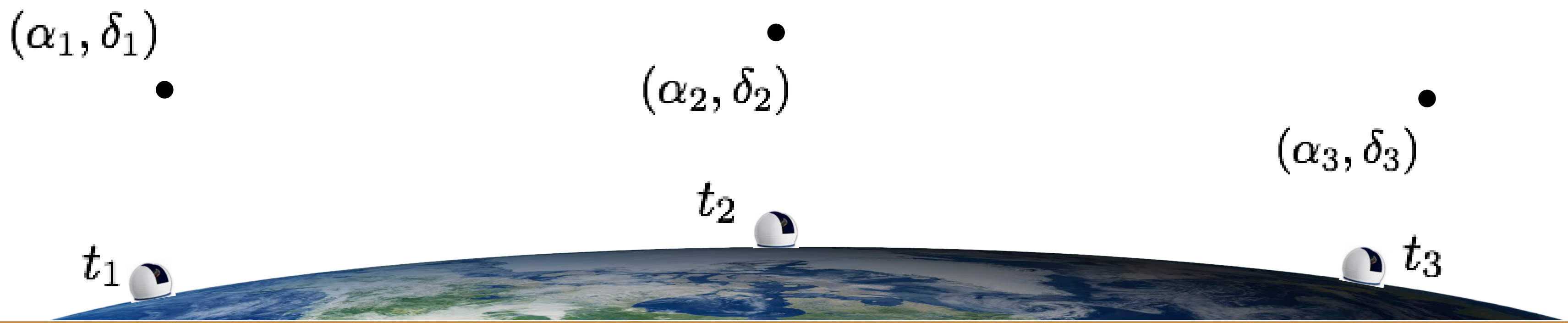
Initial Orbit Determination

- Basic idea: given three observations and associated observables
  - Find a first guess for the spacecraft state at one observation epoch
  - Propagate the state to the epochs of the remaining observations
  - Compute the predicted observables and the defects w.r.t. the true observables
  - Correct the spacecraft state to cancel the defects



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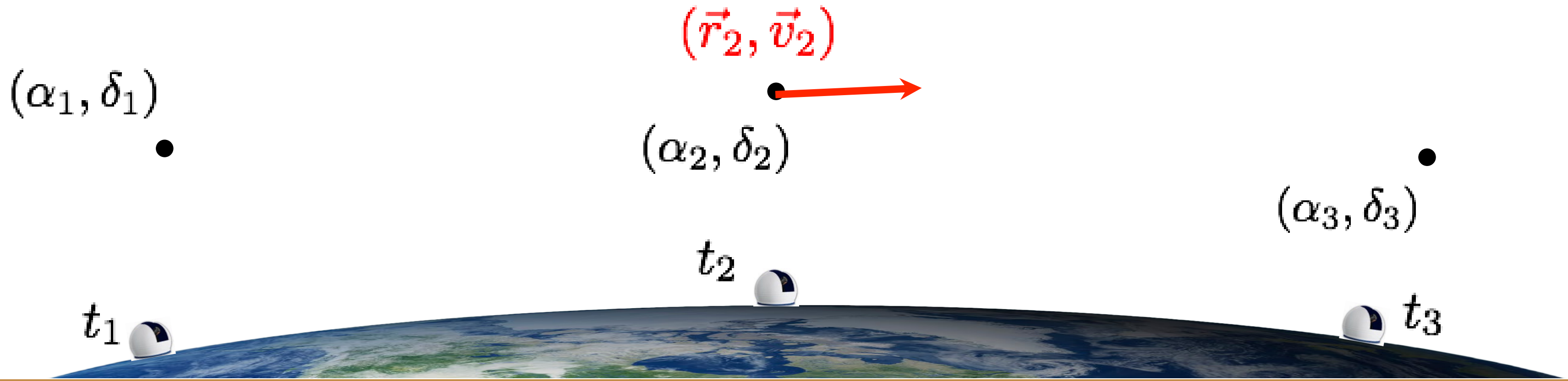

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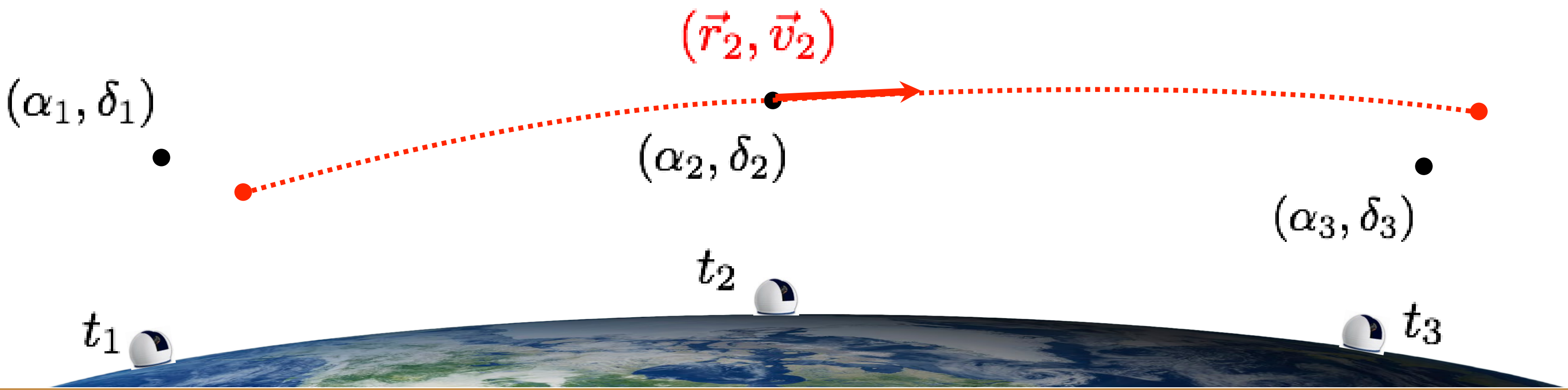

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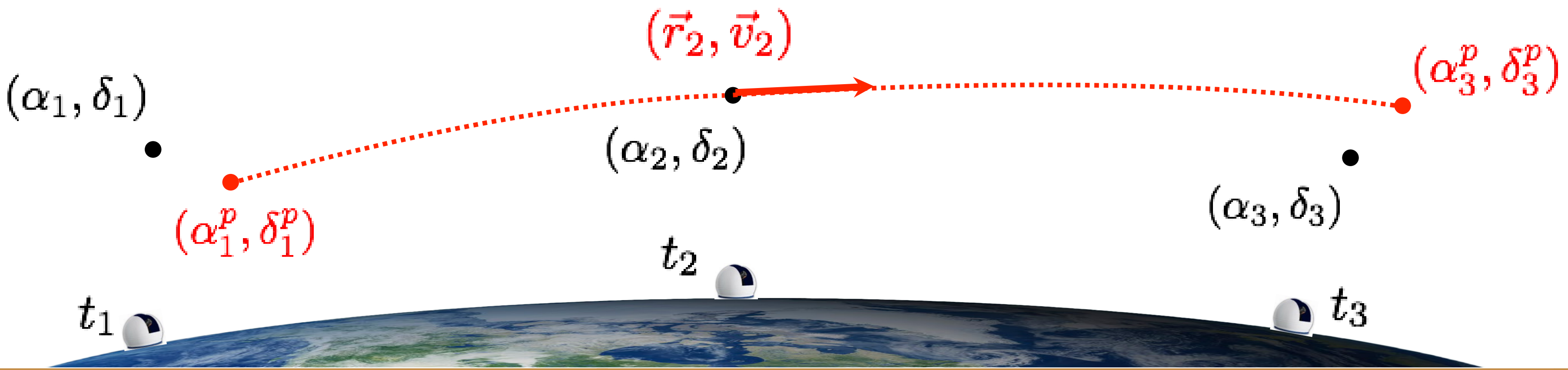

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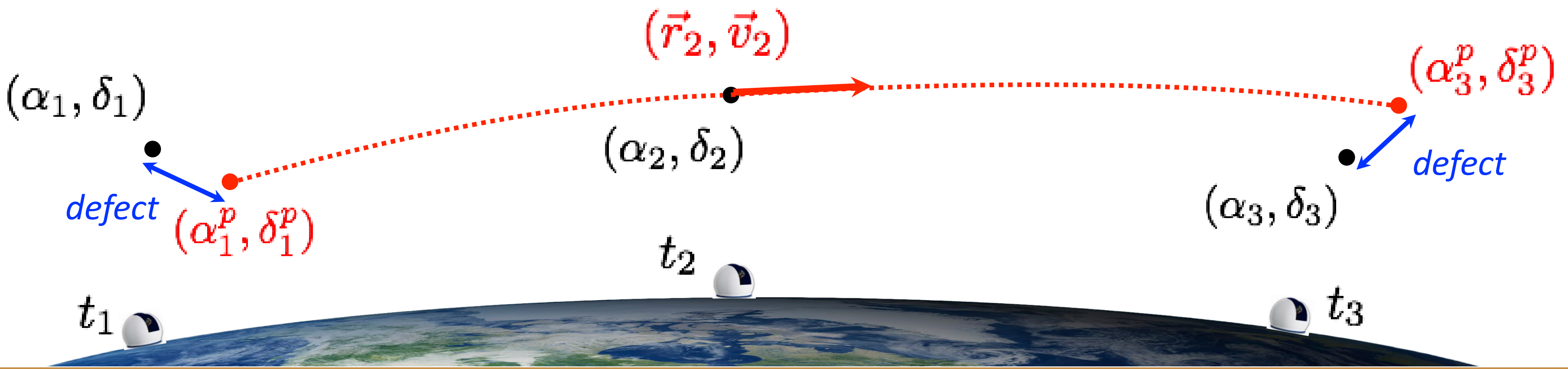

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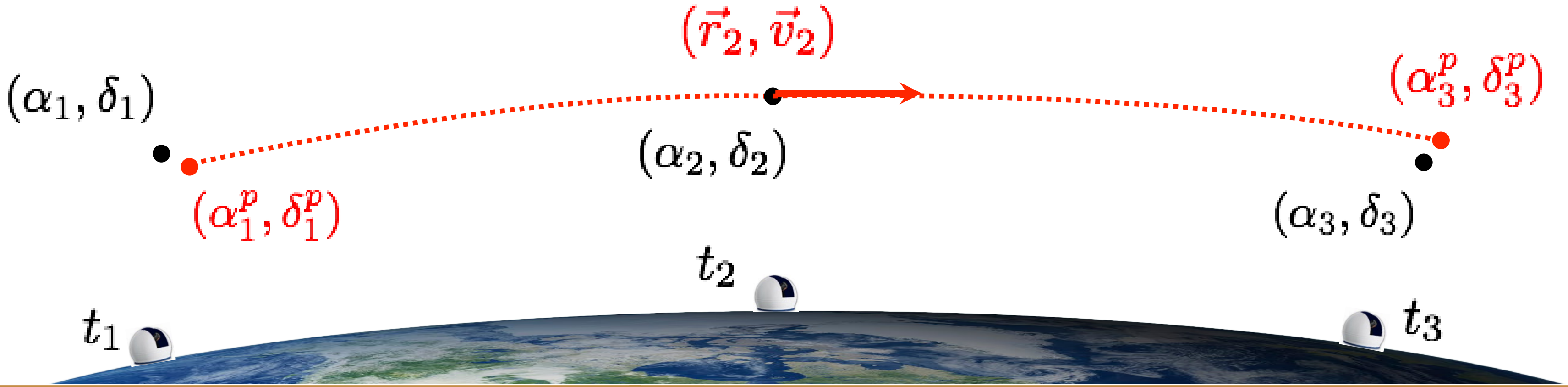

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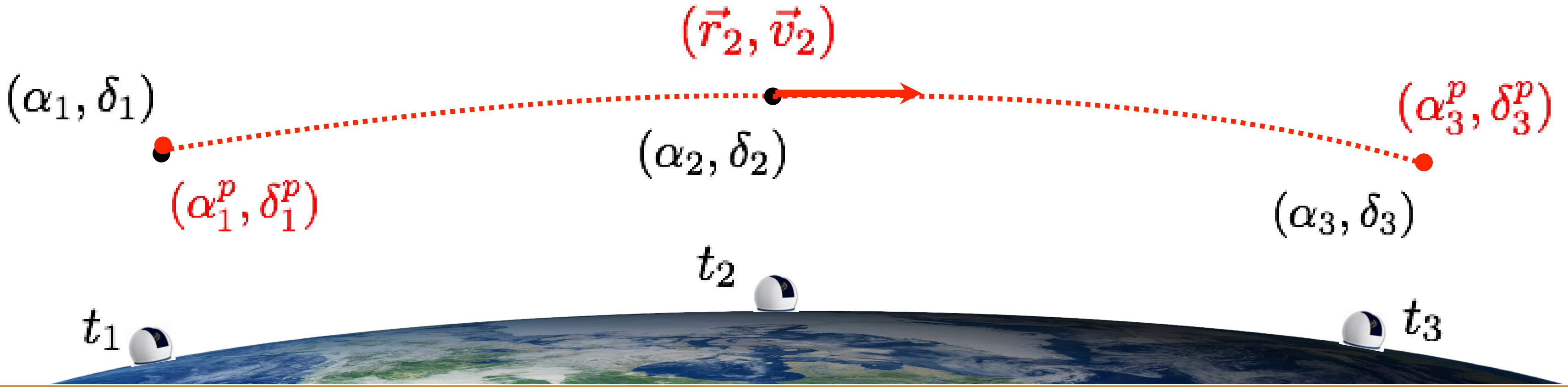

Dynamical Model

- As the measurements are performed at different epochs a function to compute the evolution of the state vector is necessary
  - Find the solution of a set of ordinary differential equations $\ddot{\boldsymbol{r}} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{a}_d$

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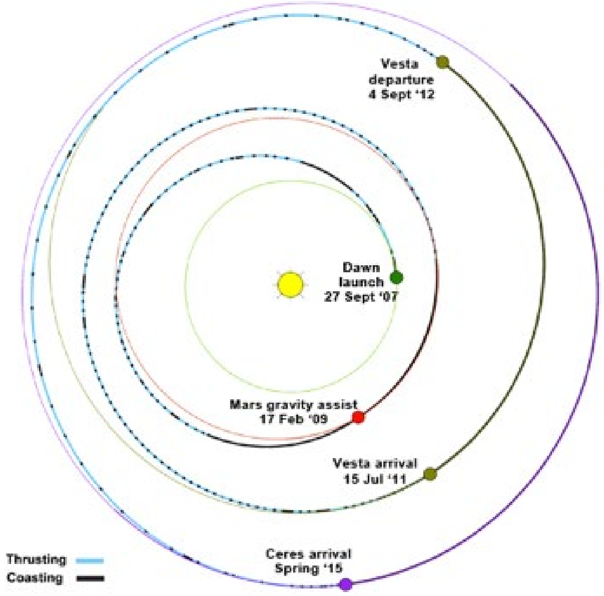
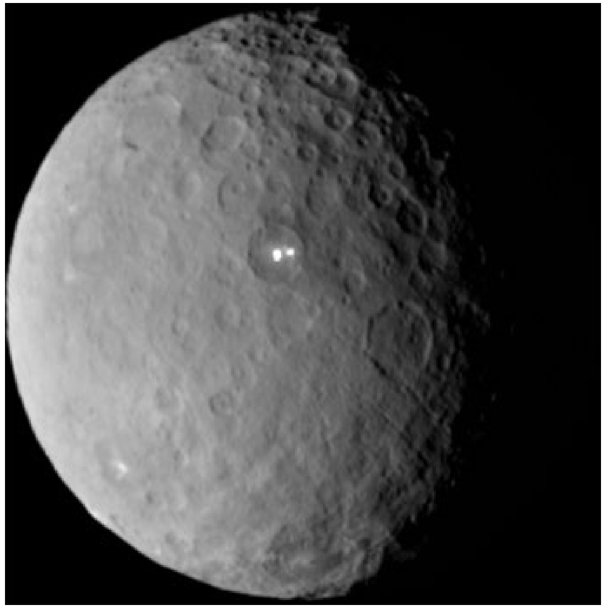
- In the case of an Earth-orbiting spacecraft $\boldsymbol{a}_d$ can include
  - Earth's gravitational harmonics
  - atmospheric drag
  - solar radiation pressure
  - third-body effect
- The solution can be found only using numerical integrators
- In initial orbit determination a two-body approximation is considered

Gauss’s Method: Some History

- Gauss’s method was derived for the "recovery" of the dwarf planet Ceres (1801):
  - Ceres orbit is between Mars and Jupiter in the main asteroid belt
  - Ceres was discovered by Father Giuseppe Piazzi 1801
  - Initially classified as a planet and later demoted an asteroid
  - Upgraded a dwarf planet in 2006, along with Pluto (downgraded)





Credits: NASA

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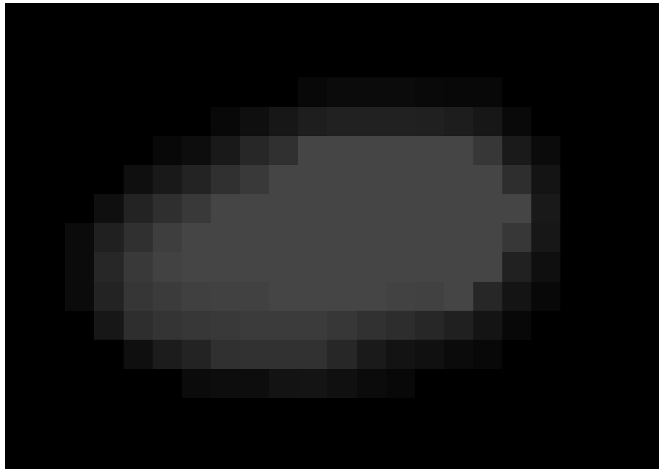
Gauss's Method: Facts

- Gauss's method:
  - is a method for initial orbit determination when three observations with telescopes are available (3 sets of right ascensions and declinations)
  - assumes two-body dynamics
  - exploits truncated Lagrangian expansions to determine a first guess solution at the epoch of the second observation
  - iterates using Kepler's dynamics until the three sets of observations are exactly satisfied in the two-body framework

After more than 200 years it is the baseline for initial orbit determination of celestial bodies and spacecraft

Effect of Uncertainties

- When it converges the Gauss method delivers a solution that exactly fits the observations in the two-body approximation
  - The observations are not exact, each measurement is affected by errors usually modeled with normal distributions
  - Dynamical perturbations deviate the object's path from two-body solution



Where is the star?

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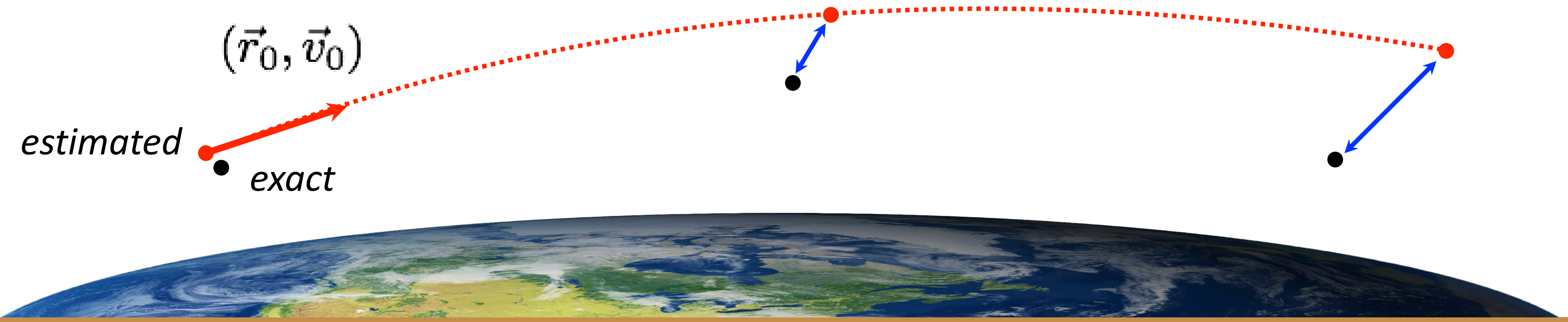
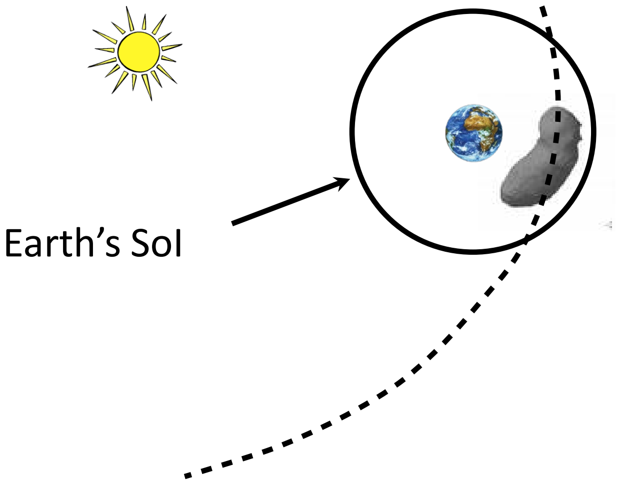
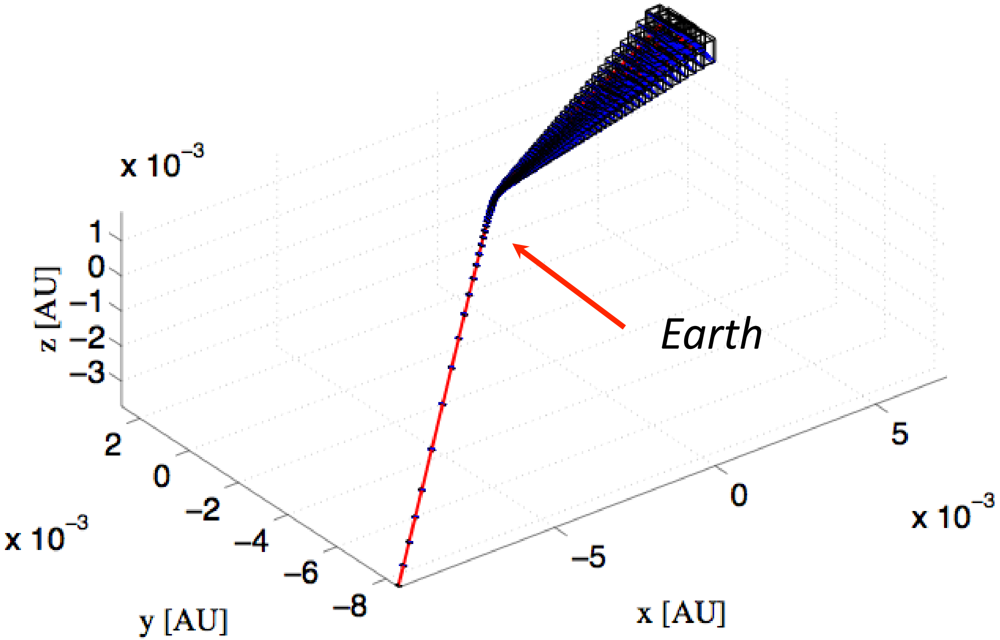


Effect of Uncertainties

- Example: Apophis encounter with Earth in April 2029





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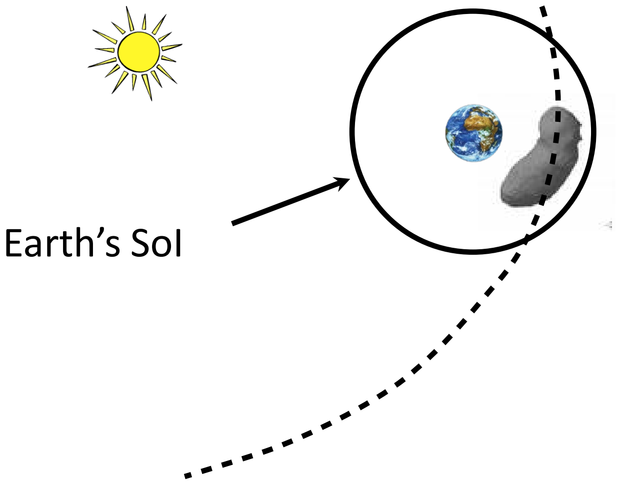
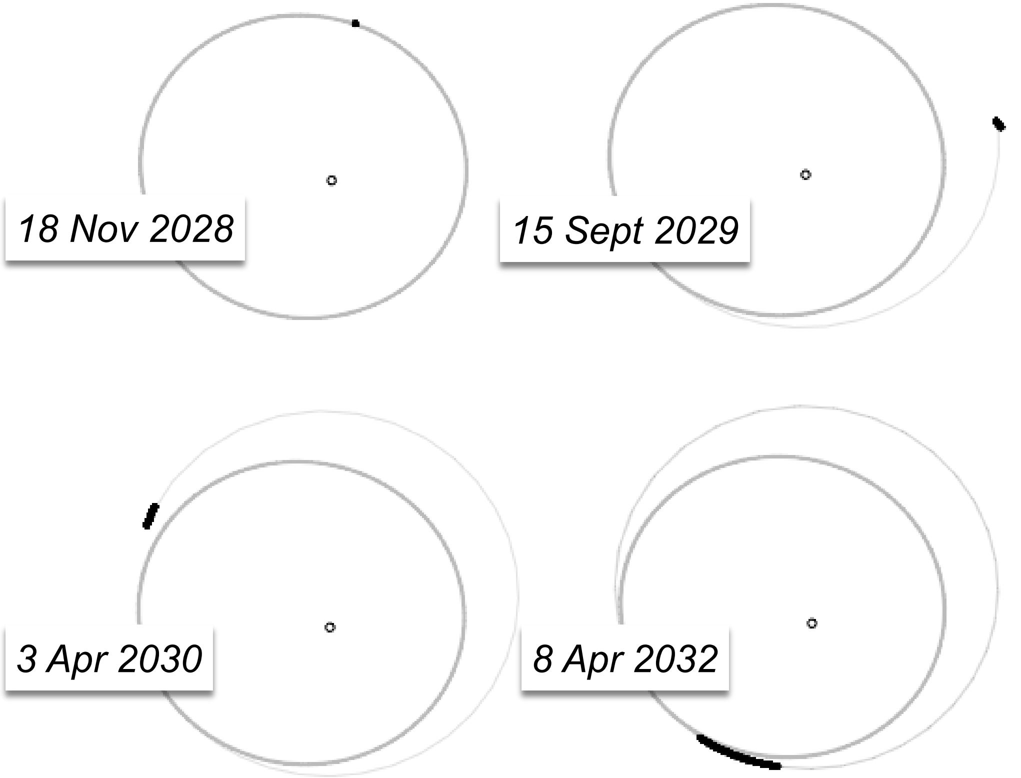


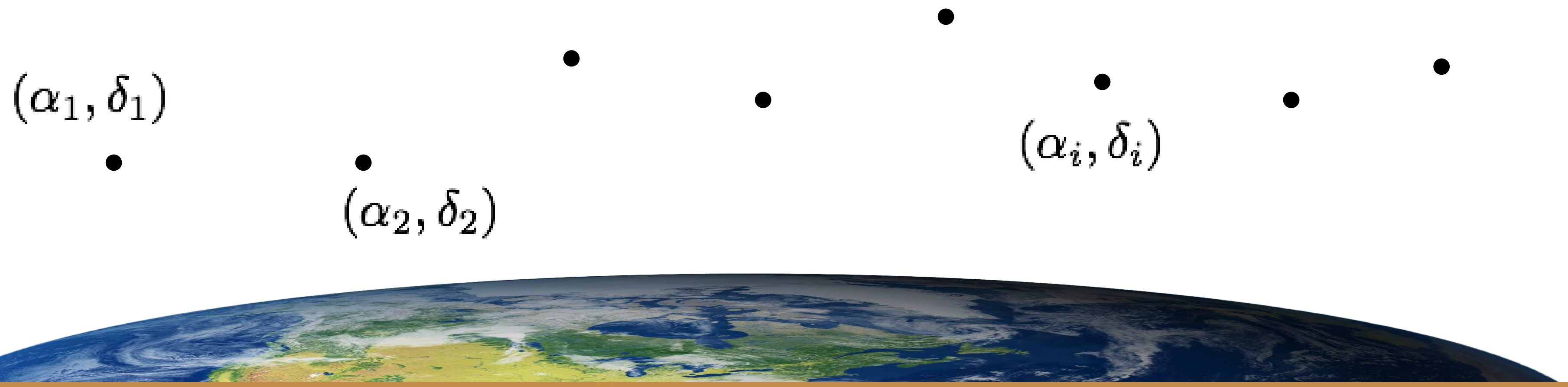


Orbit Determination

Precise Orbit Determination

Precise Orbit Determination

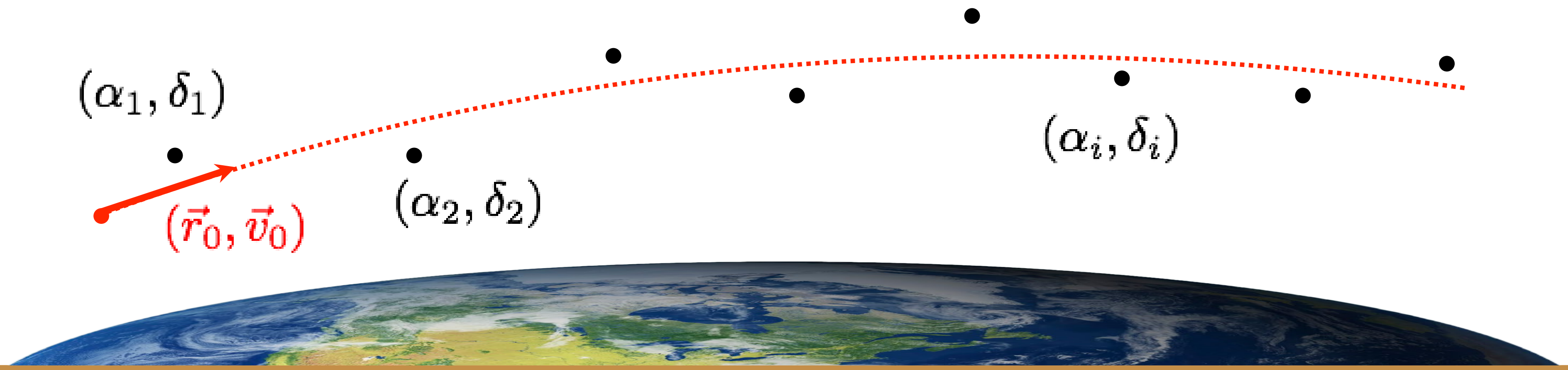
- As time goes on, several measurements become available (> 6)



Precise Orbit Determination

- As time goes on, several measurements become available (> 6)

  - The spacecraft state is found by fitting the measurements by solving a least-square problem



Least-square Problem

- Problem statement: given $\mathbf{X} = (\vec{r}_0, \vec{v}_0)$
  - Dynamics: $\dot{\mathbf{X}} = F(\mathbf{X}, t), \quad \mathbf{X}(t_0) = \mathbf{X}_0$

    dynamics *unknown*

  - Measurements: $\mathbf{Y}_i = G(\mathbf{X}_i, t_i) + \boldsymbol{\epsilon}_i; \qquad i = 1, \ldots, \ell$

    real measurements *errors*

    measurement model

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⇒ Find $\mathbf{X}_0$ that minimises $J = 1/2\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}$

vector of all errors

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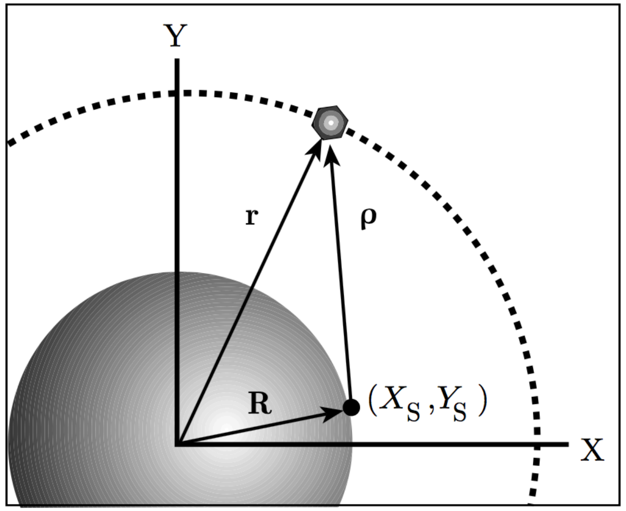
- Example:
  - Dynamics: $\ddot{\mathbf{r}} = -\dfrac{\mu \mathbf{r}}{r^3}$
  - Measurements:

$$\rho = \left[(X - X_S)^2 + (Y - Y_S)^2 \right]^{1/2} = G(\mathbf{X}_i, t_i)$$



Least-square Problem

- Linearize the equations
  - Define the displacements: $\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t), \qquad \mathbf{y}(t) = \mathbf{Y}(t) - \mathbf{Y}^*(t)$

$$\Rightarrow \begin{cases} \dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}_i = \widetilde{H}_i \mathbf{x}_i + \boldsymbol{\epsilon}_i & (i = 1, \ldots, \ell) \end{cases}$$

with $A(t) = \left[\dfrac{\partial F(t)}{\partial \mathbf{X}(t)}\right]^* \qquad \widetilde{H}_i = \left[\dfrac{\partial G}{\partial \mathbf{X}}\right]^*_i$

  - Manipulate the dynamics to relate $\mathbf{x}_i$ equations to $\mathbf{x}_0$:

$$\mathbf{x}_i = \Phi(t_i, t_0)\mathbf{x}_0$$

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State Transition Matrix

Least-square Problem

- Let's go back to the original problem:

$$\begin{cases} \dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) & \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}_i = \widetilde{H}_i \mathbf{x}_i + \boldsymbol{\epsilon}_i & (i = 1, \ldots, \ell) \end{cases} \Rightarrow \underbrace{\mathbf{x}_i = \Phi(t_i, t_0)\mathbf{x}_0}$$

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- Measurements equations are now:

$$\begin{aligned} \mathbf{y}_1 &= \widetilde{H}_1 \Phi(t_1, t_0)\mathbf{x}_0 + \boldsymbol{\epsilon}_1 \\ \mathbf{y}_2 &= \widetilde{H}_2 \Phi(t_2, t_0)\mathbf{x}_0 + \boldsymbol{\epsilon}_2 \\ &\vdots \\ \mathbf{y}_\ell &= \widetilde{H}_\ell \Phi(t_\ell, t_0)\mathbf{x}_0 + \boldsymbol{\epsilon}_\ell . \end{aligned}$$

- Define:

$$\mathbf{y} \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix}; \quad H \equiv \begin{bmatrix} \widetilde{H}_1 \Phi(t_1, t_0) \\ \vdots \\ \widetilde{H}_\ell \Phi(t_\ell, t_0) \end{bmatrix}; \quad \boldsymbol{\epsilon} \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_\ell \end{bmatrix}$$

Least-square Problem

- Rewrite measurements equations in matrix form:

$$\mathbf{y} = H\,\mathbf{x}_0 + \boldsymbol{\epsilon} \qquad (1)$$

- We can now find $\mathbf{x}_0$ that minimizes $J(\mathbf{x}_0) = \frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}$

- From (1) we have $\boldsymbol{\epsilon} = \mathbf{y} - H\,\mathbf{x}_0$

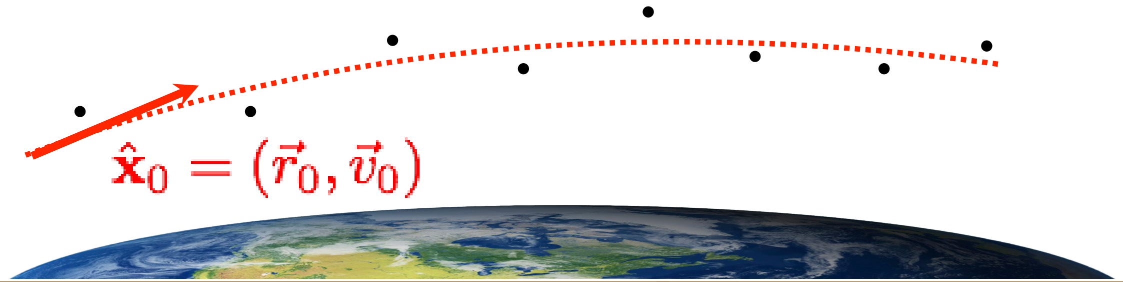
  - Minimize $\boxed{J(\mathbf{x}_0) = \frac{1}{2}(\mathbf{y} - H\,\mathbf{x}_0)^T(\mathbf{y} - H\,\mathbf{x}_0)}$

$\Rightarrow$ $\frac{\partial J}{\partial \mathbf{x}_0} = 0$ and $\frac{\partial^2 J}{\partial \mathbf{x}_0^2}$ positive definite

Least-square Problem

$$\frac{\partial J}{\partial \mathbf{x}_0} = 0 \quad \Rightarrow \quad -(\mathbf{y} - H\,\mathbf{x}_0)^T H = -H^T(\mathbf{y} - H\,\mathbf{x}_0) = 0$$

- Thus, the best estimate of $\mathbf{x}_0$ is: $\boxed{\hat{\mathbf{x}}_0 = (H^T H)^{-1} H^T \mathbf{y}}$



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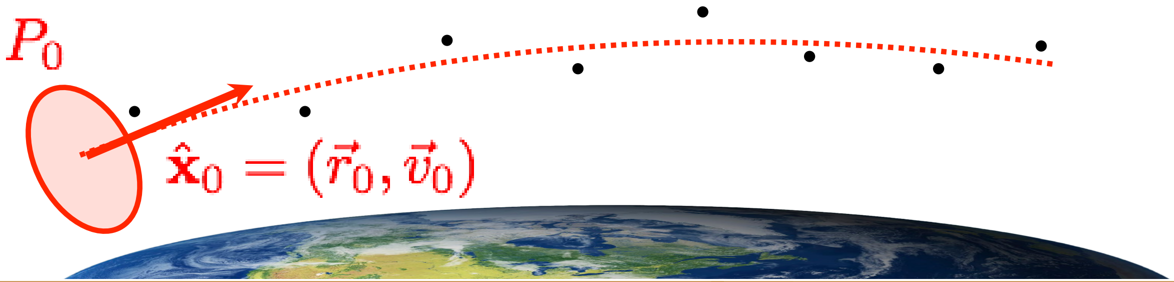

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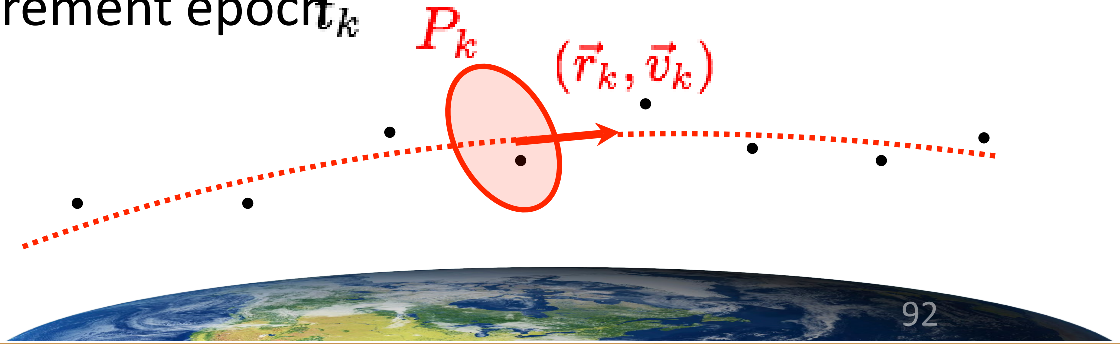
NOTE 2 : The same procedure applies if we want to obtain the estimate at any measurement epoch t_k



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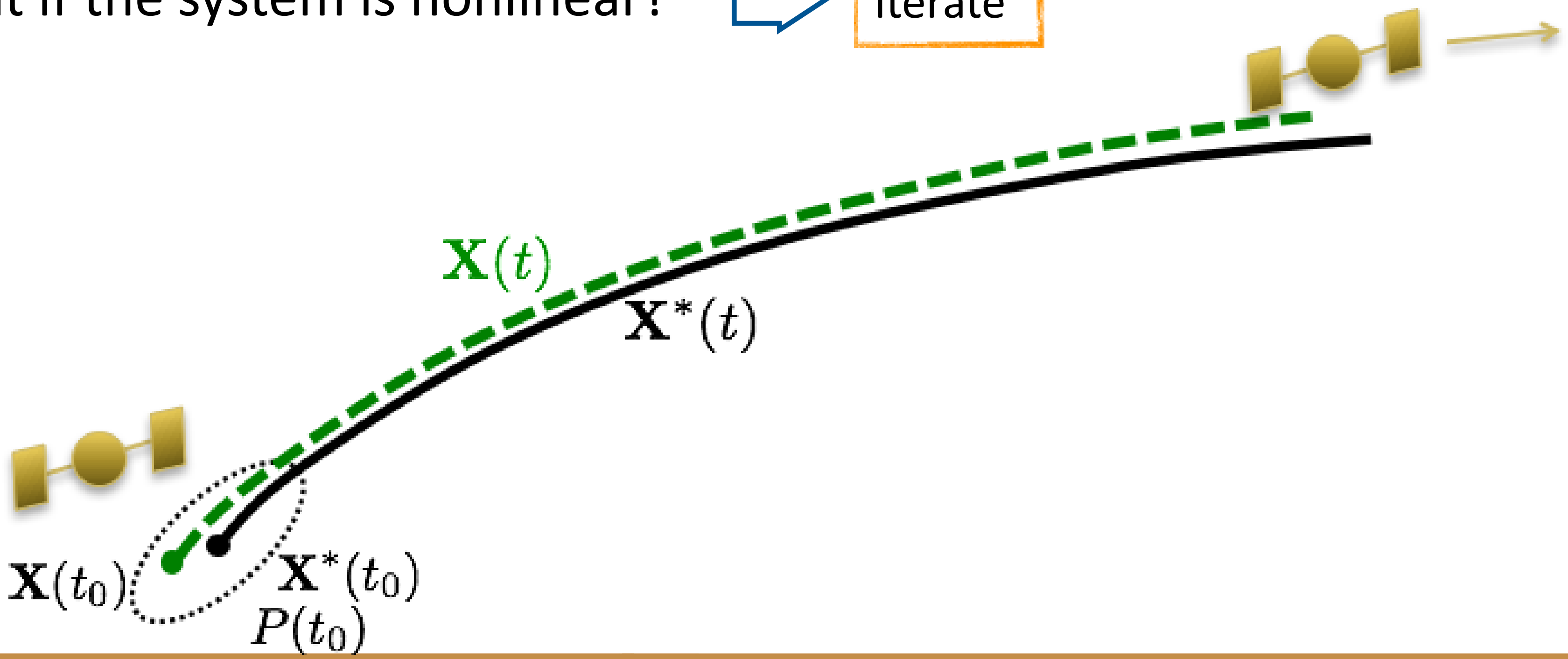

Credits: Brandon A. Jones

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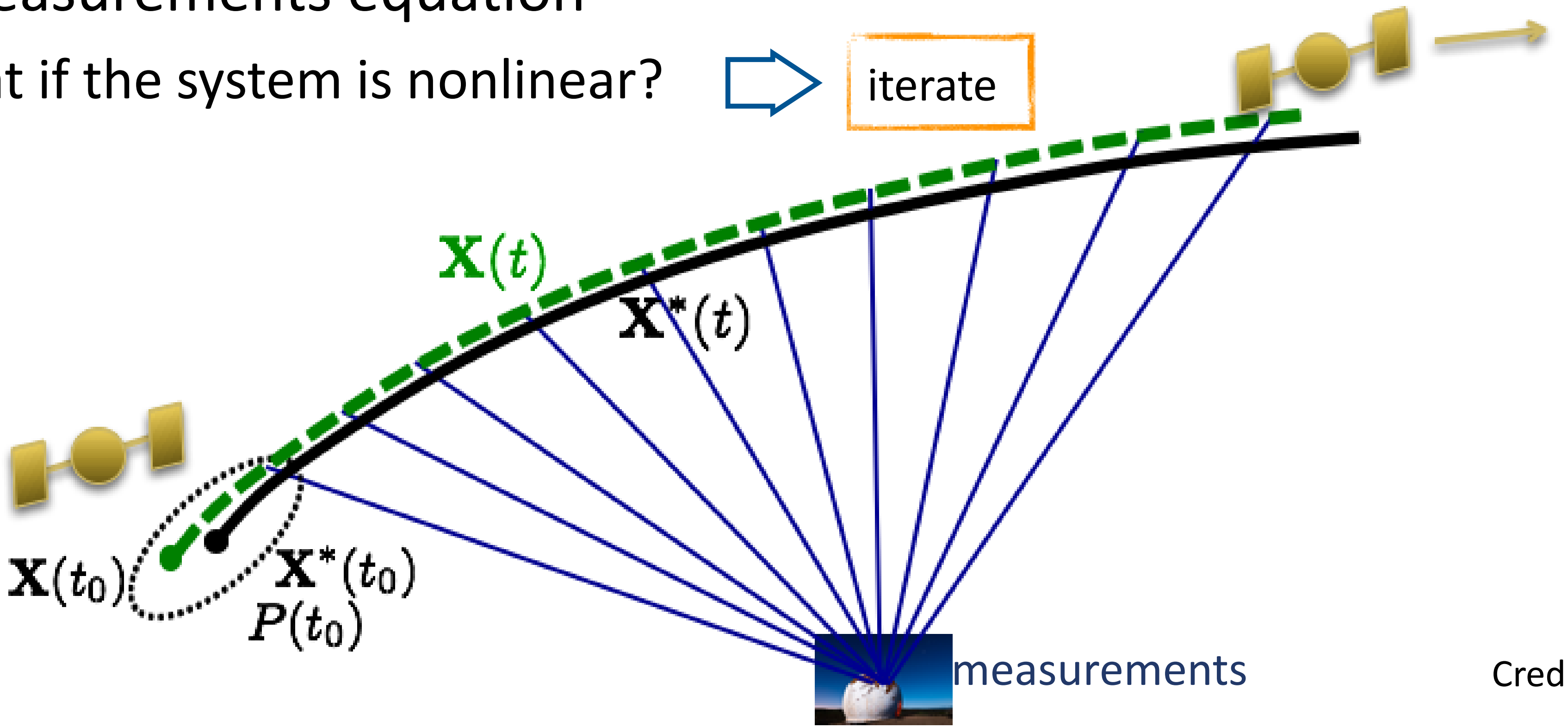

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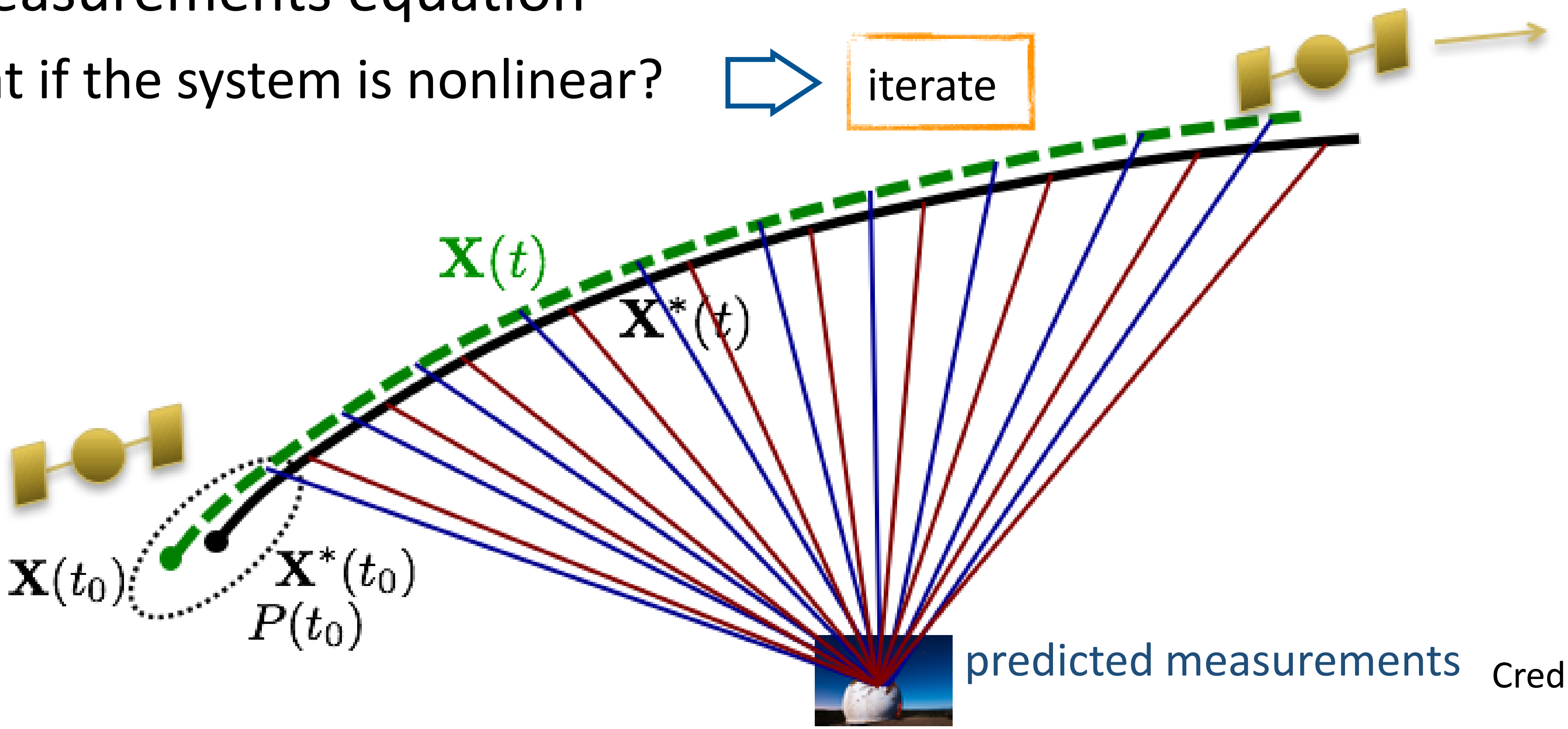

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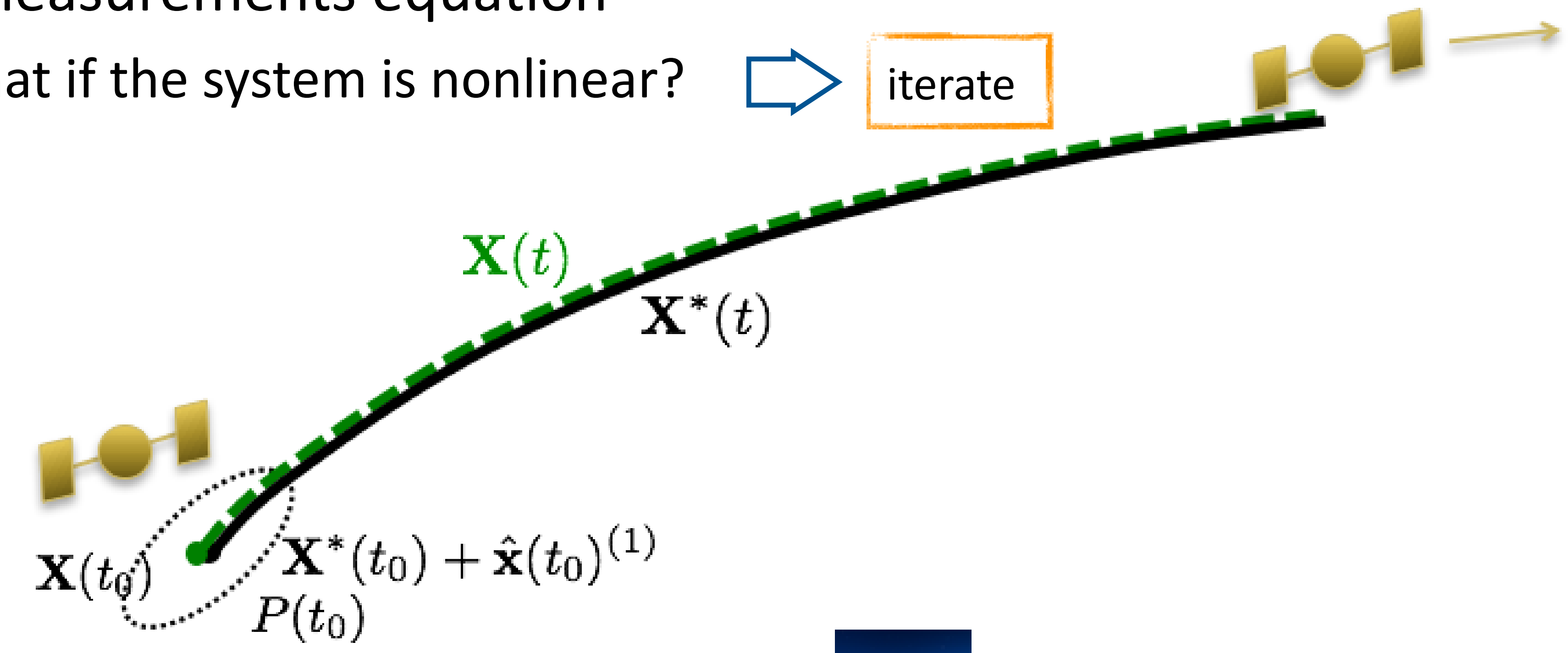

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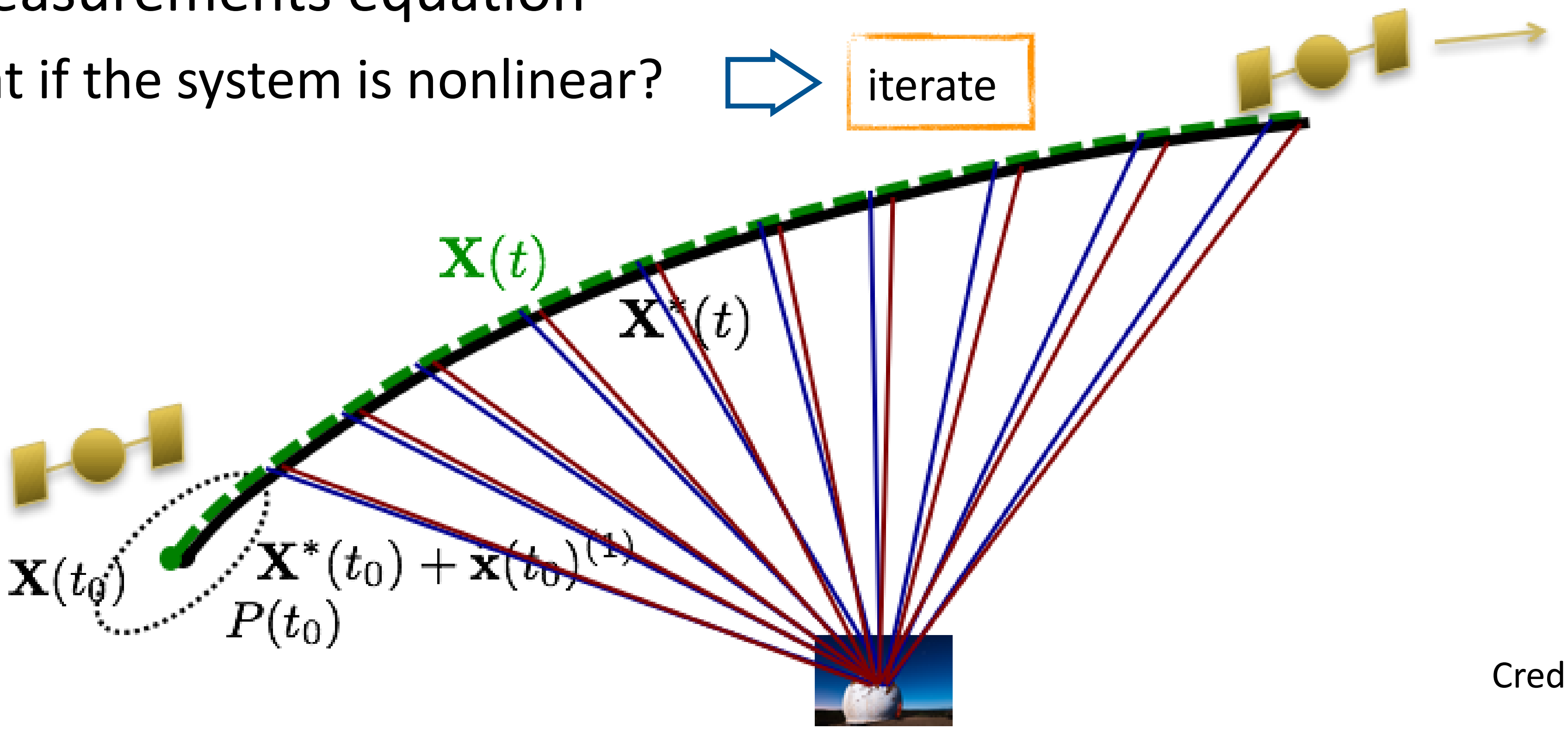

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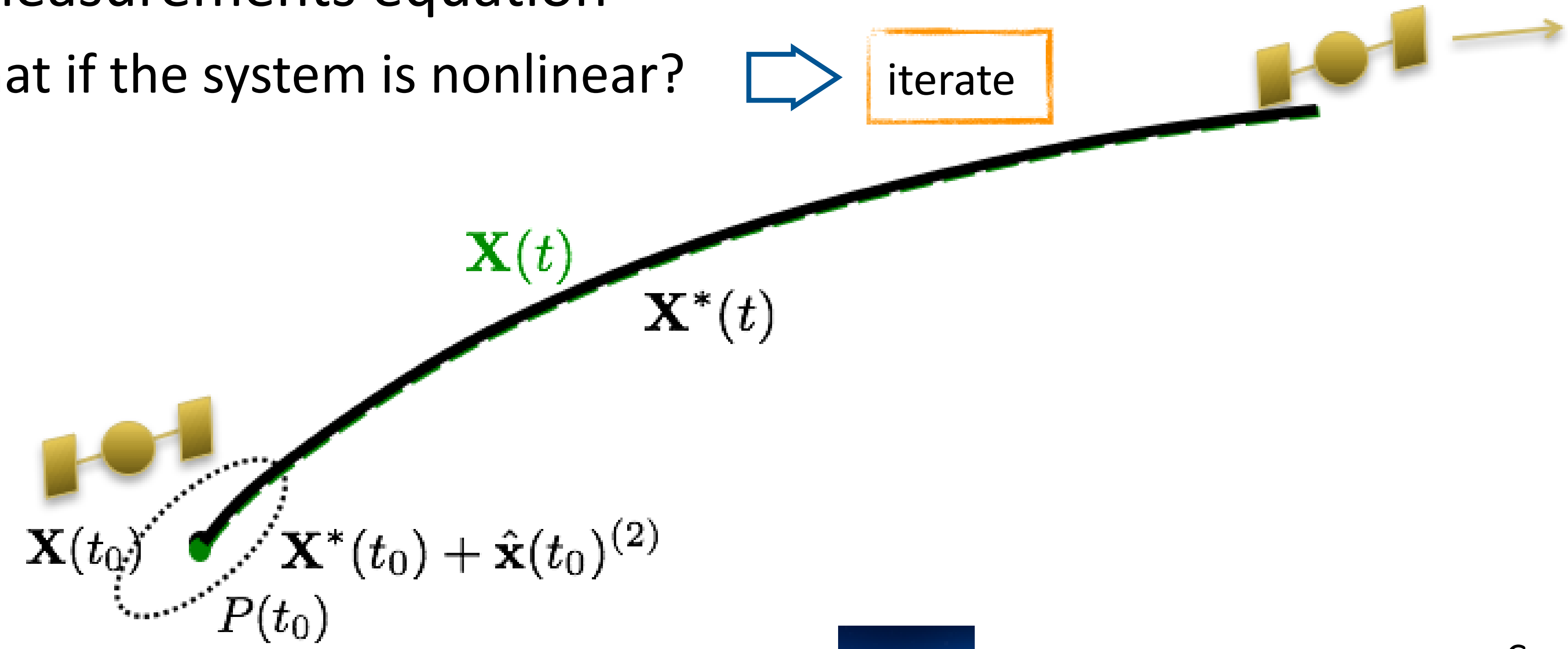

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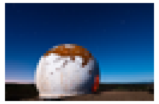
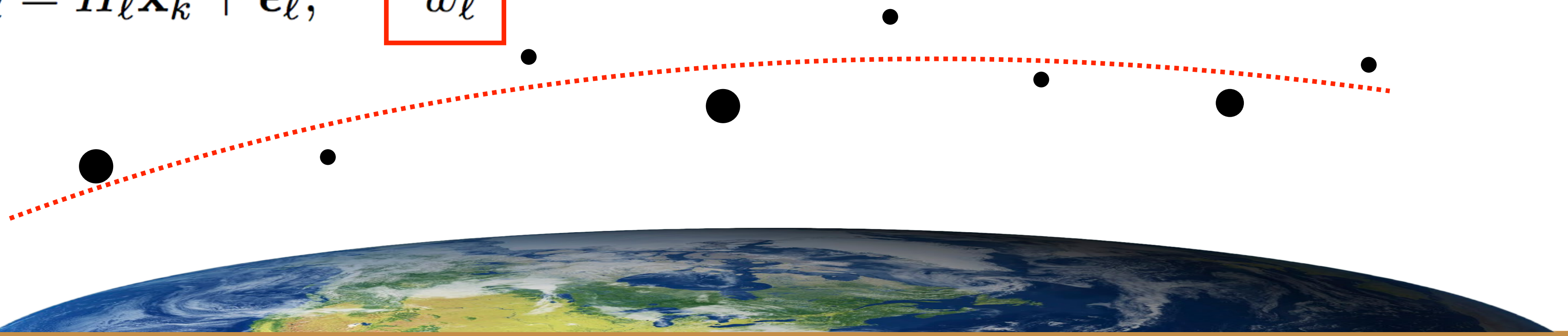


Least-square Problem

- Weighted least-square:
  - Sometimes we want to give "preference" among measurements

$$\begin{aligned}
\mathbf{y}_1 &= H_1\mathbf{x}_k + \boldsymbol{\epsilon}_1; & w_1 \\
\mathbf{y}_2 &= H_2\mathbf{x}_k + \boldsymbol{\epsilon}_2; & w_2 \\
&\vdots \quad \vdots \quad \vdots & \\
\mathbf{y}_\ell &= H_\ell\mathbf{x}_k + \boldsymbol{\epsilon}_\ell; & w_\ell
\end{aligned}$$

weights



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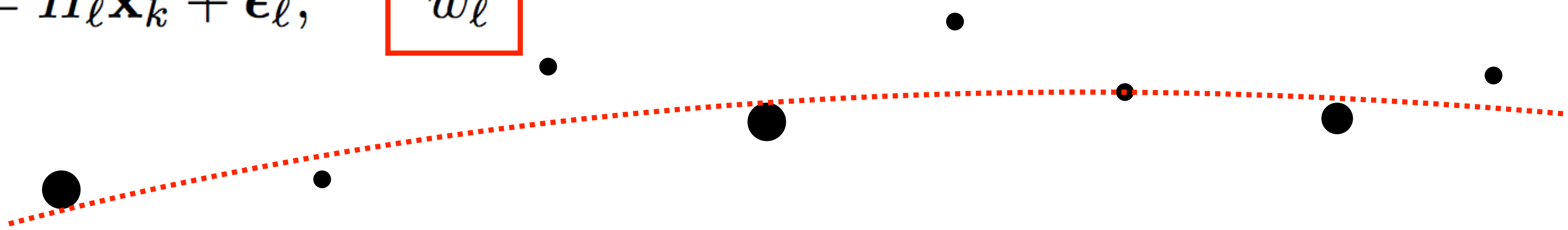
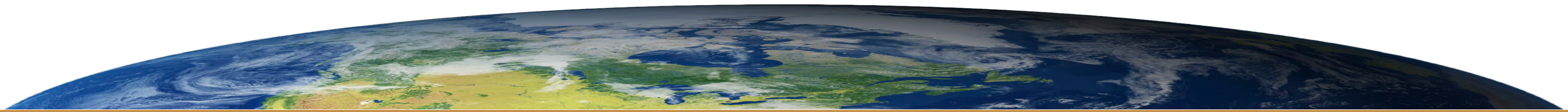
$$

weights





Least-square Problem

- Weighted least-square:
  - Sometimes we want to give "preference" among measurements

$$\begin{aligned} \mathbf{y}_1 &= H_1\mathbf{x}_k + \boldsymbol{\epsilon}_1; \quad & w_1 \\ \mathbf{y}_2 &= H_2\mathbf{x}_k + \boldsymbol{\epsilon}_2; \quad & w_2 \\ &\vdots \quad \vdots \quad \vdots & \\ \mathbf{y}_\ell &= H_\ell\mathbf{x}_k + \boldsymbol{\epsilon}_\ell; \quad & w_\ell \end{aligned}$$

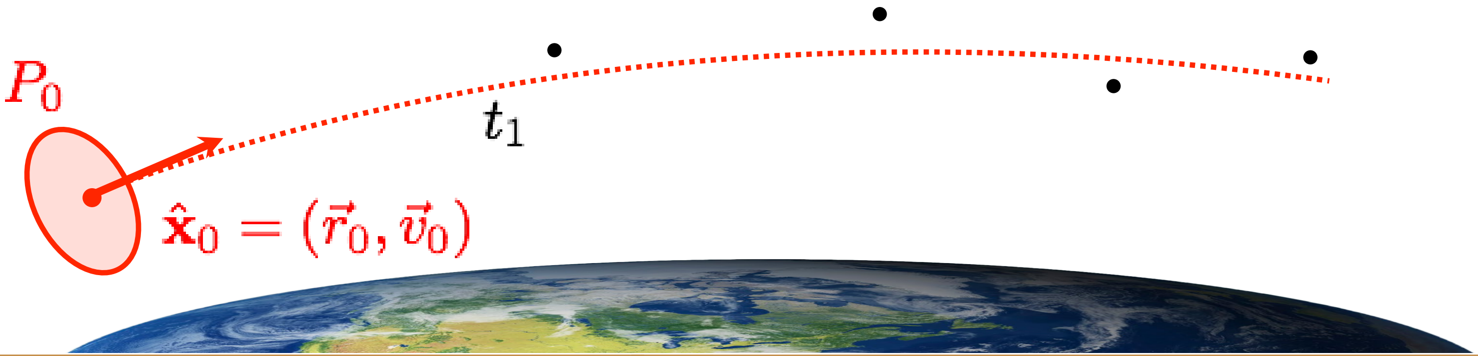
weights $\Rightarrow$

$$W = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & w_\ell \end{bmatrix}$$

  - Minimize: $J(\mathbf{x}_k) = 1/2\boldsymbol{\epsilon}^T W \boldsymbol{\epsilon}$
  - Results: the best estimate is $\boxed{\hat{\mathbf{x}}_k = (H^T W H)^{-1} H^T W \mathbf{y}}$

Kalman Filter

- Assume we have an initial estimate of mean and covariance
  - Kalman filter processes the measurements one by one as they arrive
- E.g., consider the measurements acquired at t_1
  - How can we updated the knowledge of the mean and covariance at t_1 ?



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Kalman Filter

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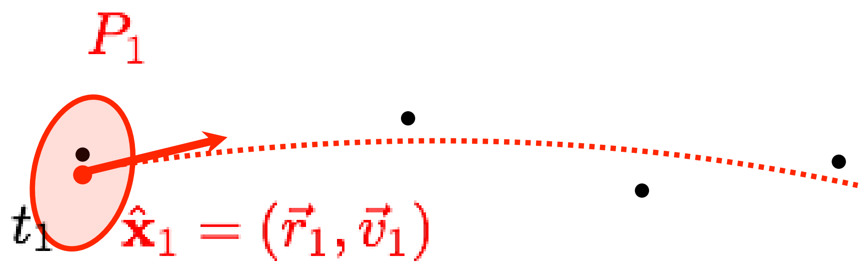

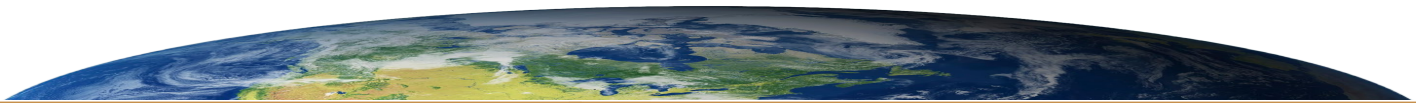

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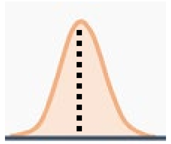
Kalman Filter

Real system

$\mathbf{y}_k$



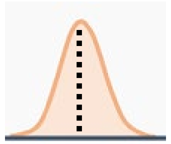
$\hat{\mathbf{x}}_{k-1}$

P_{k-1}

Kalman Filter

Real system

$\mathbf{y}_k$



$\hat{\mathbf{x}}_{k-1}$

P_{k-1}

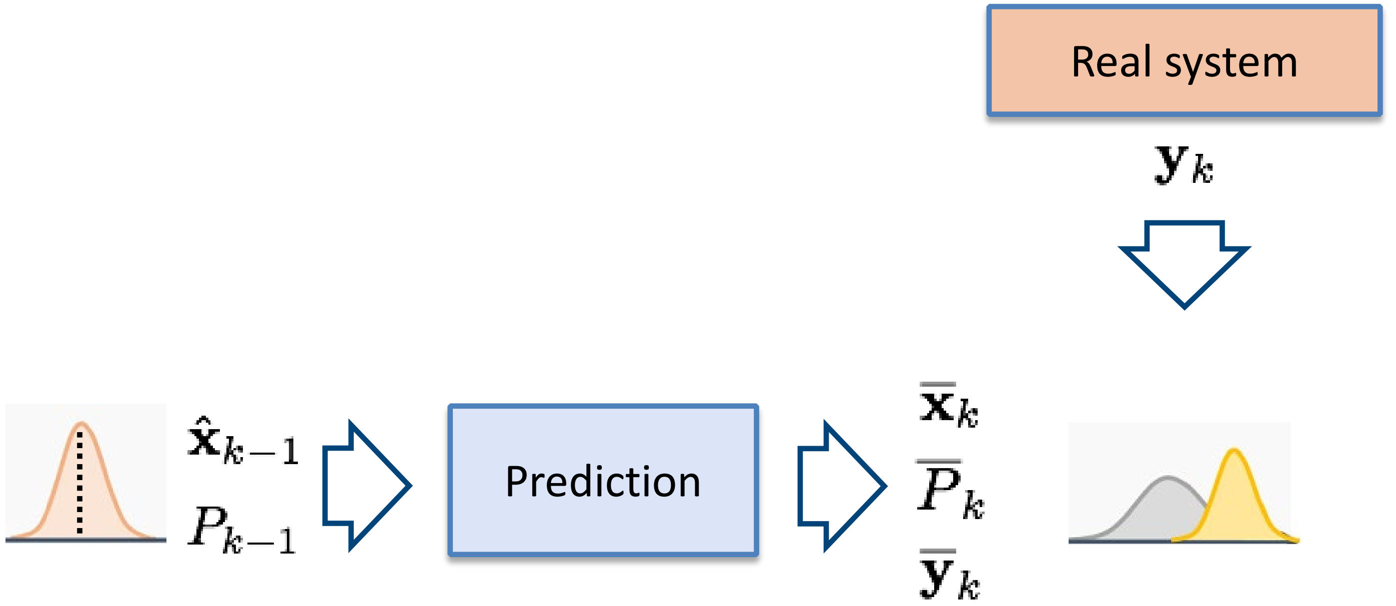
Prediction

$\overline{\mathbf{x}}_k$

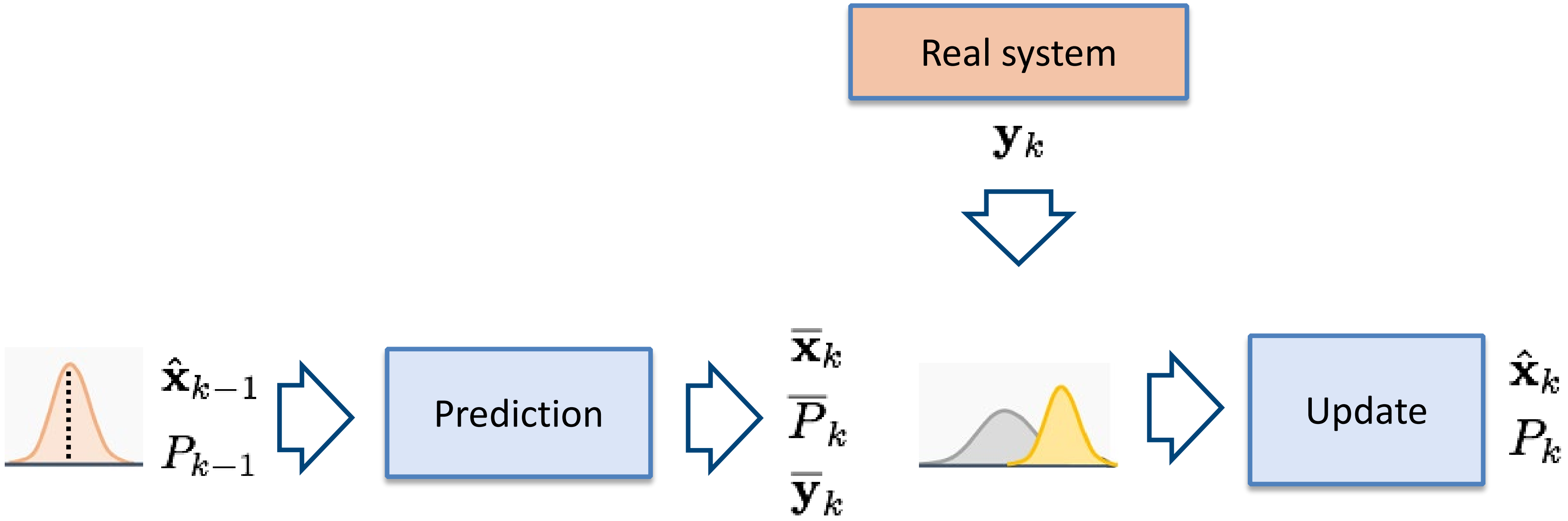
$\overline{P}_k$

$\overline{\mathbf{y}}_k$

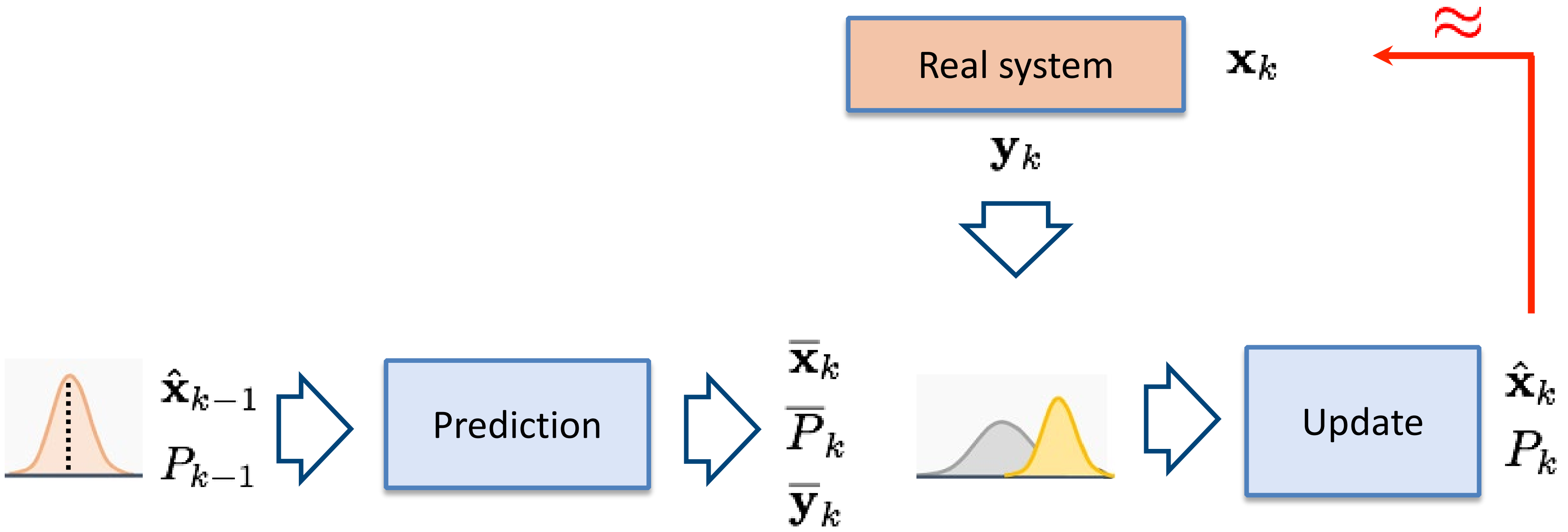
Kalman Filter



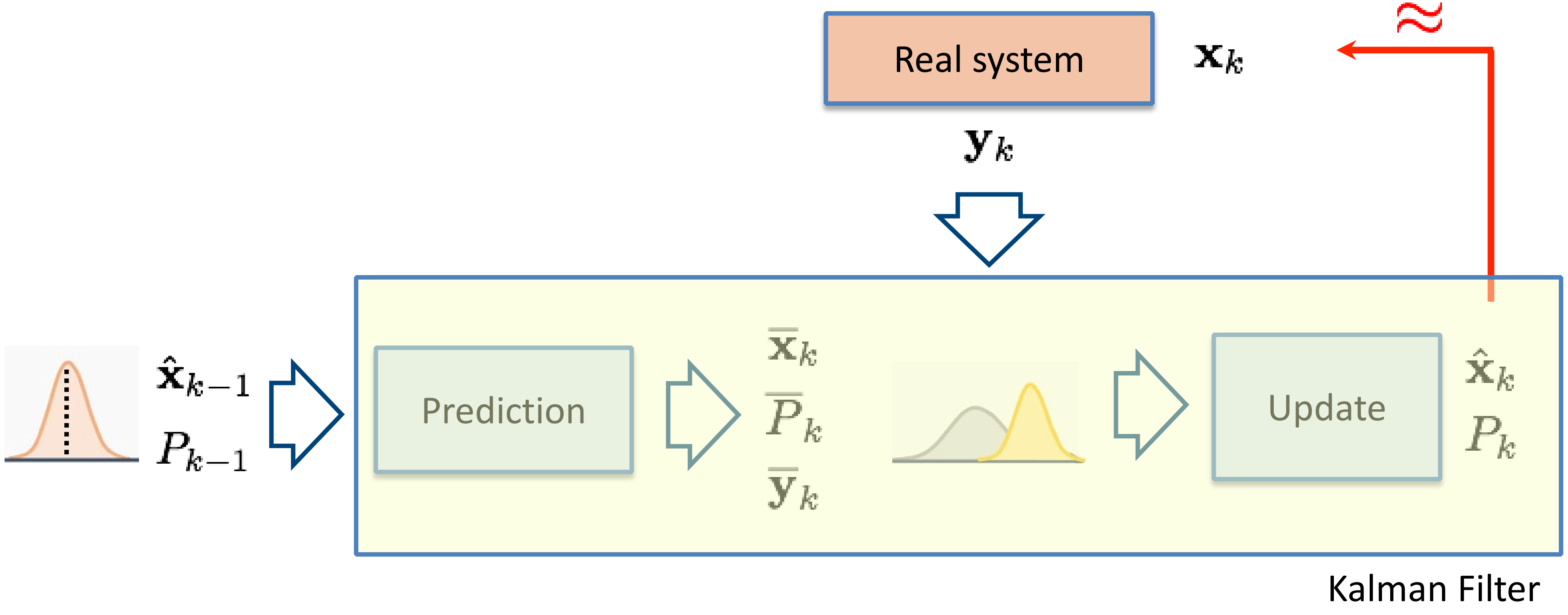
Kalman Filter



Kalman Filter



Kalman Filter



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Orbit Determination

Conclusions

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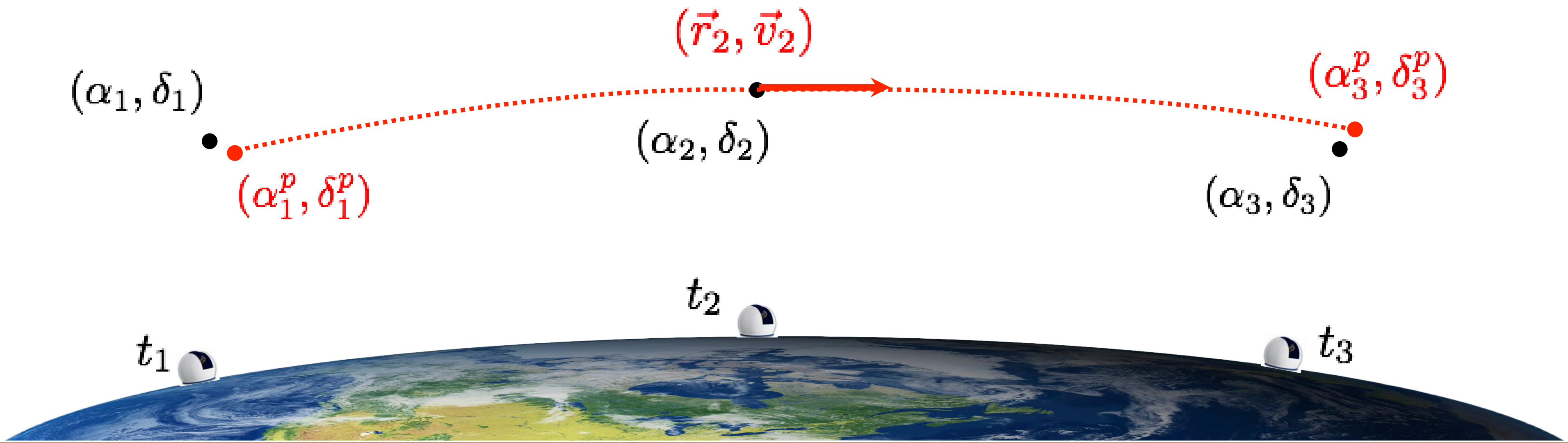
Conclusions

- If no estimates are already available ⇨ Initial orbit determination



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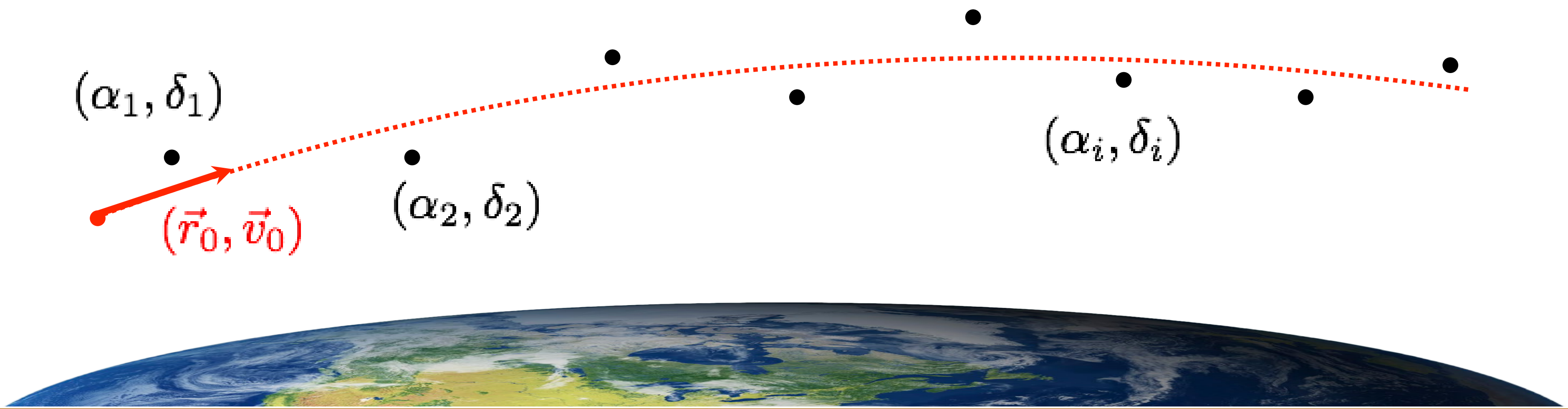
Conclusions

- If no estimates are already available ⇨ Initial orbit determination
- If an estimated state is already available ⇨ Least-squares method



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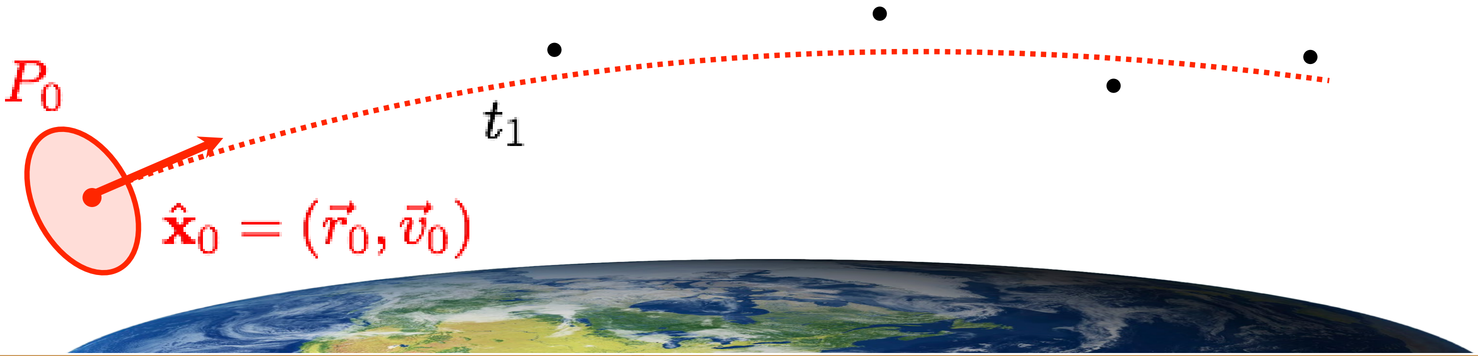
Conclusions

- If no estimates are already available ⇨ **Initial orbit determination**
- If an estimated state is already available ⇨ **Least-squares method**
- If an estimated mean and covariance is already available ⇨ **Kalman filter**



Orbit Determination

References

References

1. Schutz, Bob, Byron Tapley, and George H. Born. Statistical orbit determination. Elsevier, 2004
2. Milani, Andrea, and Giovanni Gronchi. Theory of orbit determination. Cambridge University Press, 2010
3. Crassidis, John L., and John L. Junkins. Optimal estimation of dynamic systems. CRC press, 2011
4. Beutler, Gerhard. Methods of celestial mechanics: volume I: physical, mathematical, and numerical principles. Springer Science & Business Media, 2004
5. Armellin, Roberto, and Pierluigi Di Lizia. "Probabilistic optical and radar initial orbit determination." Journal of Guidance, Control, and Dynamics 41.1 (2018): 101-118