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Symmetric solutions for the N-body problem: a computational approach

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Scientific Rationale

***N*-body problem:** predict the motion of *N* distinct particles under the action of the mutual gravitational interaction.

Poincaré conjecture: the **periodic solutions** are **dense** in the space of all possible bounded solutions.
The **chaoticity** of the *N*-body problem is one of the most famous problems in Celestial Mechanics, and the presence of (more and more complex) periodic solutions, which visit the many different regions of the phase space, goes in that direction.

Λ = class of all the possible periodic paths in the configuration space (functions from $\mathbb{R}/T\mathbb{Z}$ to a subset of $\mathbb{R}^{\dim \times N}$)
The solutions are found as **critical points** of the **Lagrangian action**

$$\mathcal{A} : \Lambda \to \mathbb{R},\ \mathcal{A}(x) = \int_0^T K(x(t)) + U(x(t))\, dt$$

***G*-equivariance**: analytical tool that can be used to find **symmetric periodic solutions** (based on invariance with respect to a group action).

Technical Objectives, Methodologies and Solutions

Making use of the G-equivariance principle, search for highly complex periodic orbits:

- ➔ start by creating a **database** of “simple” symmetric solutions, using the **SymOrb** program. SymOrb, originally written in FORTRAN, Python and GAP in the 2000s and today translated in Julia, returns symmetric solutions for the N-body problem obtained as (non necessarily minimal) critical point for the Lagrangian action. The orbit is computed starting from random initial guesses and using different numerical methods (gradient descent, Newton,...) to find critical points;
- ➔ such numerical solutions can be transformed into actual mathematical solutions by means of **computer assisted** proofs;
- ➔ the database can be enriched by computing numerically, for each solution, **stability indicators** like the Floquet exponents (to estimate the rate of separation of close orbits) or the Morse index (to estimate the variation in action for close orbits);
- ➔ the orbits in the database can be **clustered** in terms of their shape and stability similarity (in Fourier space), to group together the ones with similar properties;
- ➔ starting from the above database, one can use optimization algorithms to produce **new orbits,** not necessarily periodic (for example, as combinations of the ones already found).

Technical Objectives, Methodologies and Solutions

Example of database for 3 bodies in $\mathbb{R}^2$ and $\mathbb{R}^3$

In this case, there are more than 100 different symmetries. For every of them, one can start with different initial data and different methods to find critical points. The construction of the database is divided into two steps:

➔ **first set**: *minimal* critical trajectories (already done, over 1000 numerically computed solutions have been found);

➔ **second set**: critical trajectories which are *not necessarily minimal* (using the results already obtained with step 1 to optimize the process; at present, over 2000 solutions have been found);

➔ **stability analysis**: once the solutions are found, their stability indicators can be numerically computed with suitable routines.

Note that we look for **collisional** orbits as well.

Main Results

- ➔ **SymOrb routine:** originally written in Fortran, Python and GAP, it has been translated in Julia and an optimised interface has been added, along with visualization tools (code in an online repository)

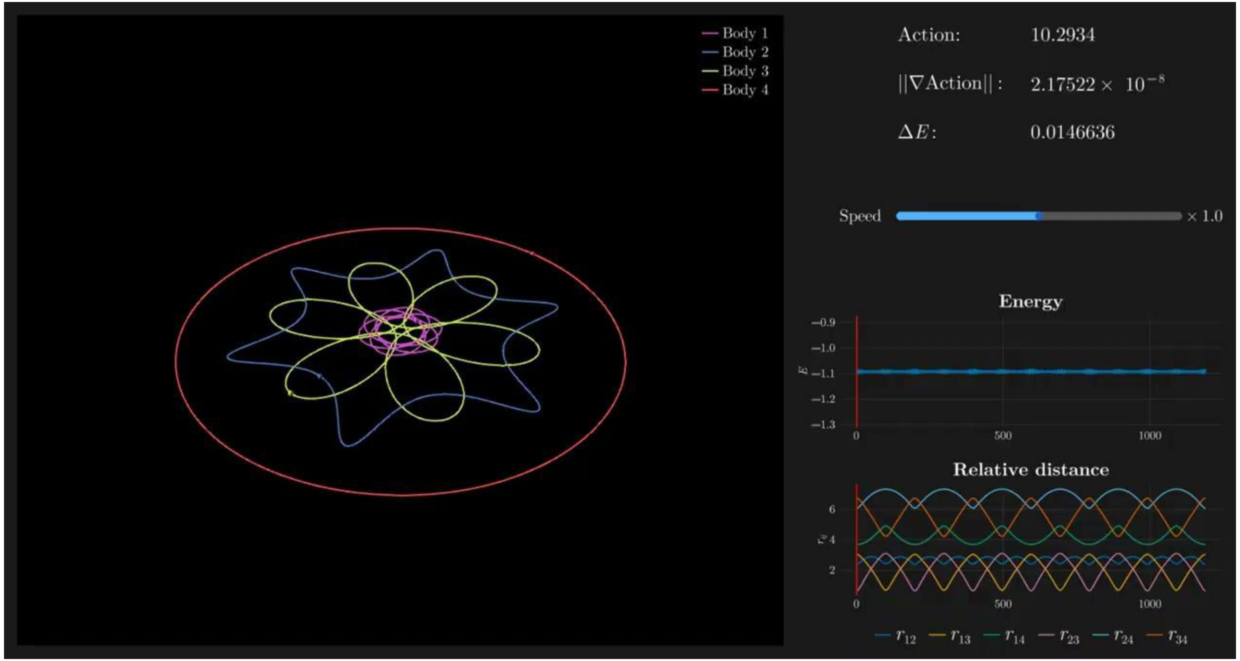
- ➔ **Theoretical framework:** starting from the existing literature on the *G*-equivariant orbith of the *N*-body problem (*Ferrario - Terracini, 2004*), a detailed documentation on the theoretical framework has been written, and it has been the subject of a PhD course held in Spring 2023. After that, an extensive study on how to construct new, non trivial symmetry groups has been carried on (preprint submitted).

- ➔ **Database:** an extended database, has been produced for 3 bodies in 2 and 3 dimensions (see next slides). The database of step 1 had to be rebuilt after correcting a bug in the code which increased a lot the computational time (database in an online repository).

- ➔ **Stability indicators:** the routines for the numerical computations of stability indicators have been created (from scratch). They have been tested on several orbits and optimised. The results obtained numerically have been compared with the existing literature (preprint in preparation).

Main Results

Examples of orbits from the database ~ 400 different trajectories



Main Results

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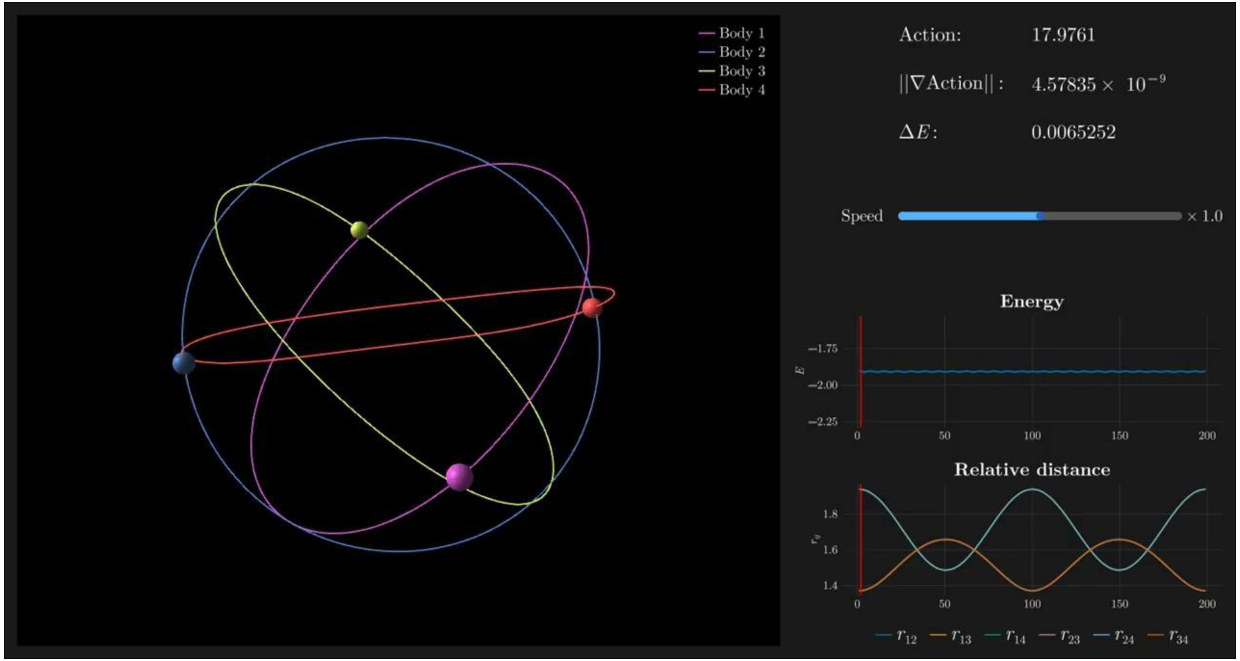
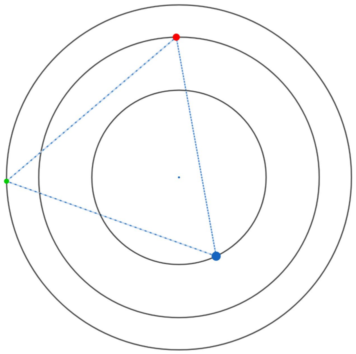
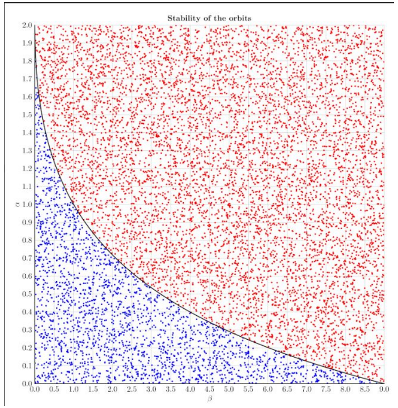


Accomplished Work, Results

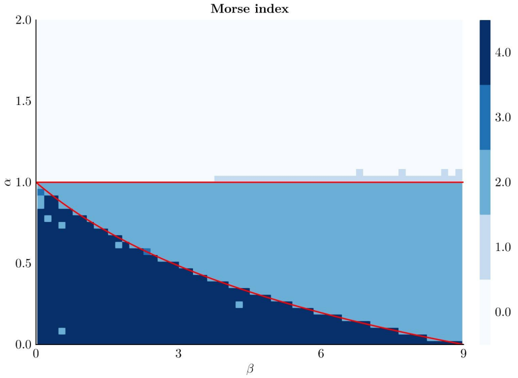
Stability indicators: distinguish stable from unstable solutions.
Compute them analytically is particularly difficult, and it has been done only for few types of solutions. For random mass combinations, we compared the expected theoretical results to numerical ones.



Case study: triangular Lagrange solutions



Stability through Floquet exponents



Stability through Morse indices

NB: the application of this routine on a large database is already an interesting by-product of the project.

Final steps

- ➔ **Database:** the database can be further enriched by considering a greater number of bodies in 2 or 3 dimensions, which could correspond to taking more and more complex symmetry groups. For such periodic trajectories, the stability properties can be derived.

- ➔ **Computer assisted proofs:** to prove rigorously the existence of solutions, starting from numerically computed ones.

- ➔ **Clustering:** the wide database can be refined by grouping together the orbits with similar dynamic properties.

- ➔ **New complex orbits:** search for strategies to explore the database and combine different periodic symmetric solutions, to the end of creating new solutions that "shadow" more than one simple trajectory, passing from one to another.

- ➔ **Parametrise the chaos:** find solutions that visits many different regions of the phase space. This is the ultimate objective of our analysis, since theoretically the chaotic behaviour is related to the presence of a solution dense in the phase space; such trajectory can be found as a limit of complex periodic solutions.