# Causal nature and dynamics of trapping horizon in black hole collapse

***Ilia Musco***
(University of Geneva / University of Barcelona)

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Collaborators:
*John Miller* (Oxford)
*Alexis Helou* (Paris)

*"I must say I find it brave of the authors to invest so much time and effort in recreating numerical models that were thoroughly investigated 50 years ago, with little prospect of discovering anything new, but their approach to the problem is fresh and interesting."*

The Anonymous Referee

# Introduction

- **Spherically symmetric** metric in comoving coordinates with ***t*** "cosmic time":

$$ds^2 = a^2(r,t)dt^2 + b^2(r,t)dr^2 + R^2(r,t)d\Omega^2$$

- **Proper time** and **proper distance** operators:

$$D_t \equiv \frac{1}{a}\frac{\partial}{\partial t} \;\Rightarrow\; U \equiv D_t R \qquad\qquad D_r \equiv \frac{1}{b}\frac{\partial}{\partial r} \;\Rightarrow\; \Gamma \equiv D_r R$$

- **Perfect Fluid:** $T^{\mu\nu} = (e+p)u^\mu u^\nu + p g^{\mu\nu}$

- Constraint equation (integrating $G_{00}$) : $\boxed{\Gamma^2 = 1 + U^2 - \frac{2M}{R}}$

- Mister-Sharp Mass : $M = \int 4\pi e R^2 dR$

$$D_t M = -4\pi p R^2 U$$

# Trapping Horizons

Expansion of **ingoing/outgoing** null-rays :

$$k^a / l^a = \left(\frac{1}{a}, \pm\frac{1}{b}, 0, 0\right) \implies \theta_\pm = h^{cd}\nabla_c k_d = \frac{2}{R}(U \pm \Gamma)$$

$$h_{ab} = g_{ab} + \frac{1}{2}(k_a l_b + l_a k_b) \qquad\qquad k^a l_a = -2$$

**Black Hole / Cosmological horizon :** $\theta_\pm = 0 \;\Rightarrow\; \left.\frac{1}{a}\frac{dR}{dt}\right|_\pm = 0 \;\Rightarrow\; \Gamma^2 = U^2$

$$R(r,t) = 2M(r,t)$$

**The horizon condition is independent of the slicing and holds also within a non-vacuum moving medium**

The so-called **apparent horizon** of a black hole (which is a future trapping horizon) is the **outermost trapped surface for outgoing radial null rays** while the **trapping horizon for an expanding universe** (which is a past trapping horizon) **is foliated by the innermost anti-trapped surfaces for ingoing radial null rays**.

# Causal Nature

$$\alpha \equiv \frac{\mathcal{L}_v \theta_v}{\mathcal{L}_{nv}\theta_v} \quad \begin{cases} \alpha > 0 & : \text{ space-like} \\ \alpha = 0 / \infty & : \text{ null} \\ \alpha < 0 & : \text{ time-like} \end{cases}$$

Lie Derivatives:

$$\begin{cases} \mathcal{L}_+ \theta_v = \mathcal{L}_k \theta_v = k^a \partial_a \theta_v = \left( \frac{1}{a}\frac{\partial}{\partial t} + \frac{1}{b}\frac{\partial}{\partial r} \right) \theta_v \\ \mathcal{L}_- \theta_v = \mathcal{L}_l \theta_v = l^a \partial_a \theta_v = \left( \frac{1}{a}\frac{\partial}{\partial t} - \frac{1}{b}\frac{\partial}{\partial r} \right) \theta_v \end{cases}$$

$$\mathcal{L}_\pm \theta_v = (D_t \pm D_r)\,\theta_v$$

$$\boxed{\alpha = \left. \frac{4\pi R^2 (e+p)}{1 - 4\pi R^2 (e-p)} \right|_H}$$

# Horizon Velocity

3-velocity of the horizon with respect the matter: $v_H \equiv \left( \frac{b}{a} \frac{dr}{dt} \right)_H$

$$\theta_v = 0 \quad \Rightarrow \quad D_t \theta_v + \frac{b}{a}\frac{dr}{dt} D_r \theta_v = 0$$

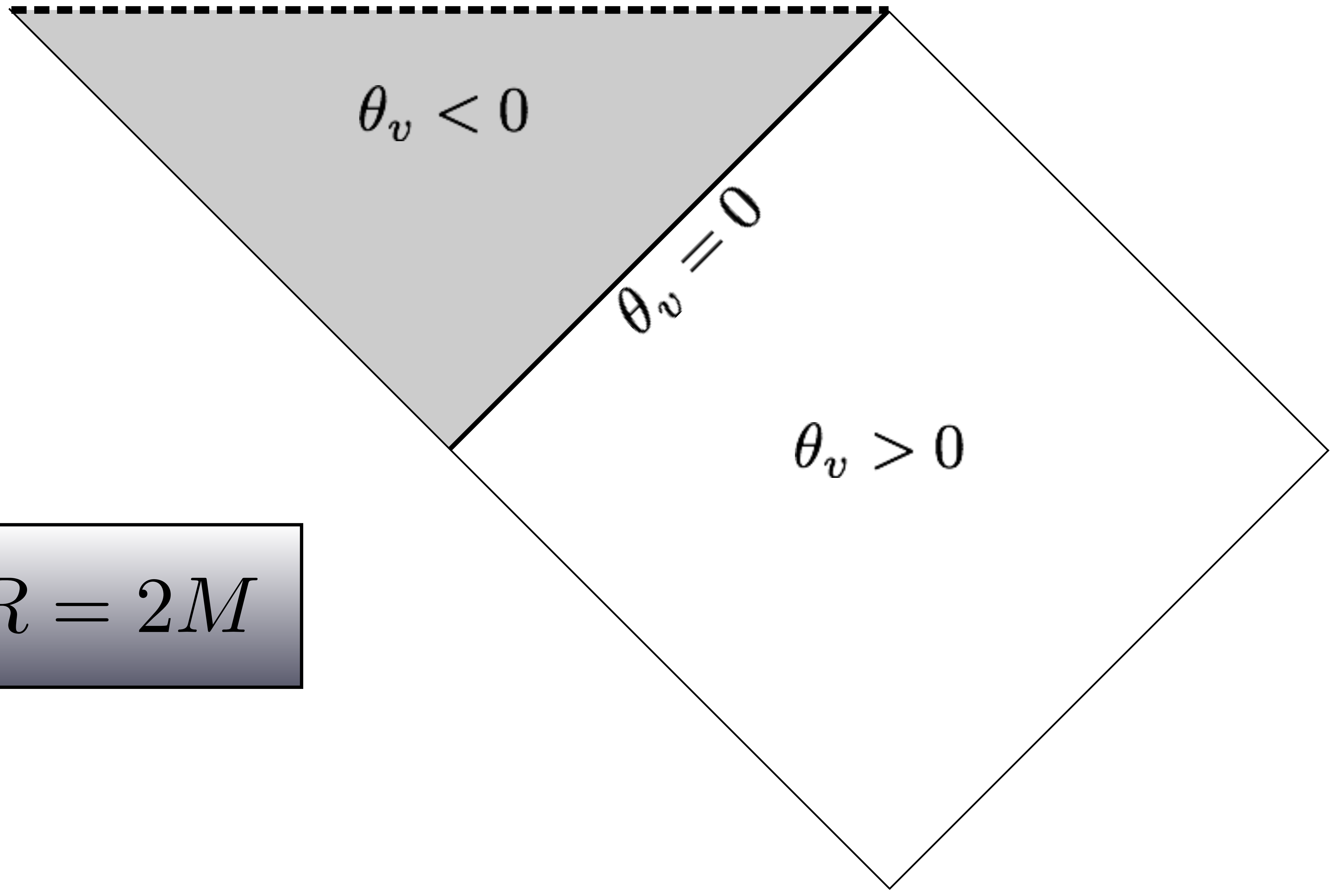
$$v_H \equiv -\frac{D_t \theta_v}{D_r \theta_v} \quad \Rightarrow \quad v_H = - \left. \frac{D_t \left( \Gamma^2 - U^2 \right)}{D_r \left( \Gamma^2 - U^2 \right)} \right|_H$$

$$v_H = - \left. \frac{\mathcal{L}_+ \theta_v + \mathcal{L}_- \theta_v}{\mathcal{L}_+ \theta_v - \mathcal{L}_- \theta_v} \right|_H \quad \Rightarrow \quad v_H = \pm \frac{1+\alpha}{1-\alpha}$$

$$\boxed{v_H = - \left. \frac{U}{\Gamma} \right|_H \left. \frac{1 + 8\pi R^2 p}{1 - 8\pi R^2 e} \right|_H}$$

$$\begin{cases} |v_H| > 1: \text{ space-like} \\ |v_H| = 1: \text{ null} \\ |v_H| < 1: \text{ time-like} \end{cases}$$

# Schwarzschild Black Hole space-time




$$R = 2M$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \qquad T_{\mu\nu} = (e+p)u_\mu u_\nu - p g_{\mu\nu}$$

## COSMIC TIME

$$ds^2 = -a^2\,dt^2 + b^2\,dr^2 + R^2 d\Omega^2$$

$$D_t \equiv \frac{1}{a}\left(\frac{\partial}{\partial t}\right) \qquad D_r \equiv \frac{1}{b}\left(\frac{\partial}{\partial r}\right)$$

- Proper time / space derivative

$$U \equiv D_t R \qquad \Gamma \equiv D_r R$$

- 4-velocity & Lorentz factor

$$D_t U = -\left[\frac{\Gamma}{(e+p)} D_r p + \frac{M}{R^2} + 4\pi R p\right]$$

- Euler equation

$$D_t \rho = -\frac{\rho}{\Gamma R^2} D_r (R^2 U)$$

- Continuity equation

$$D_t e = \frac{e+p}{\rho} D_t \rho$$

$$D_t M = -4\pi R^2 p U$$

- Mass conservation

$$dM = aU\,dt + b\Gamma\,dr$$

$$D_r a = -\frac{a}{e+p} D_r p$$

- Lapse equation / pressure gradients

$$D_r M = 4\pi R^2 \Gamma e$$

$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

- Constraint equation

# Equation of State

energy density: $e = \rho(1+\epsilon)$

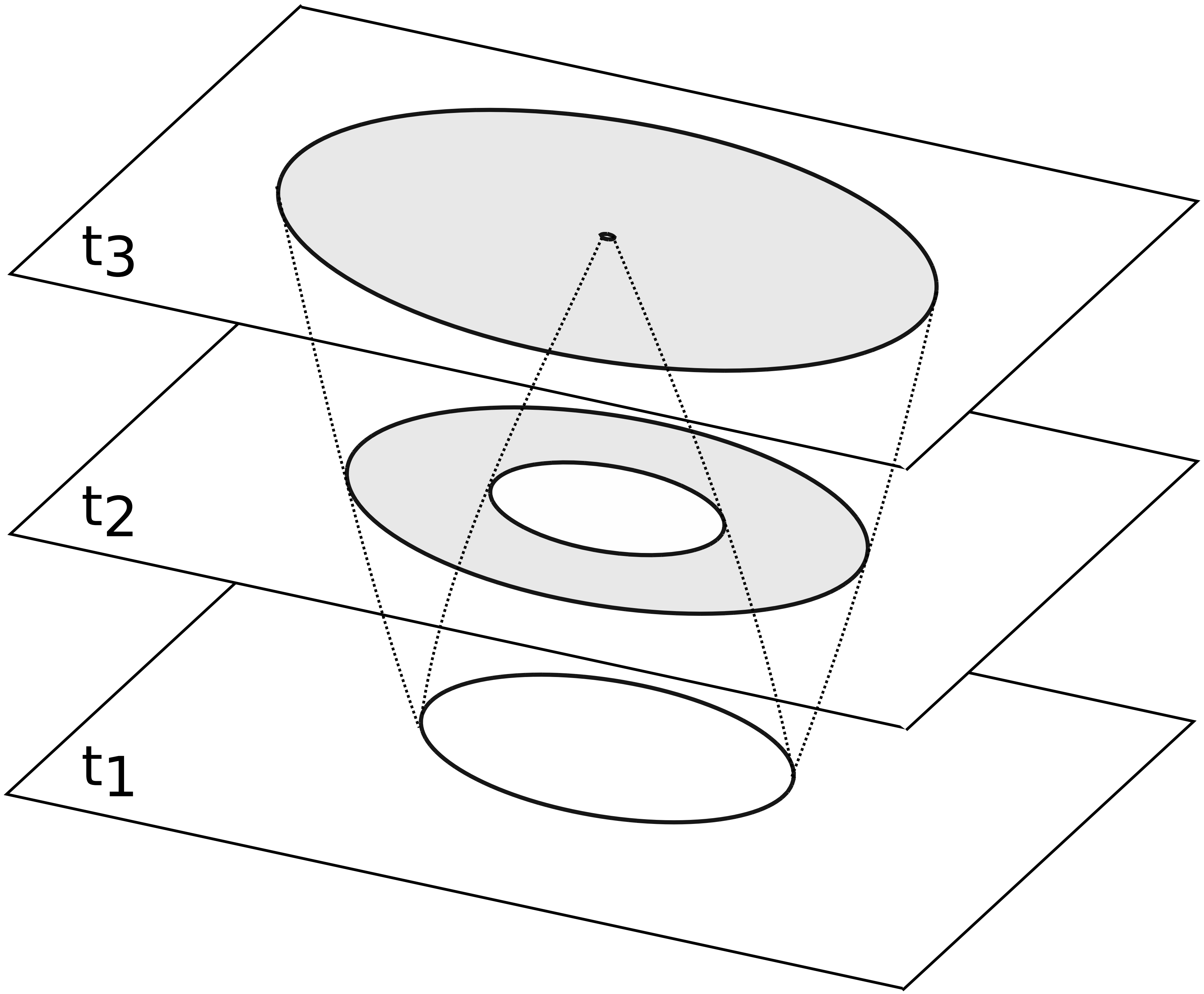
pressure: $p = (\gamma - 1)\rho\epsilon$

- $\rho$ → rest mass density
- $\epsilon$ → specific internal energy (velocity dispersion)
- $\gamma$ → adiabatic index - particle degree of freedom

- Barotropic fluid (no rest mass density): $p = we$ with $w \in [0,1]$
    - radiation dominated era: $w = 1/3$ RADIATION $(\gamma = 4/3)$
    - matter dominated era: $w = 0$ DUST $(\gamma = 1)$

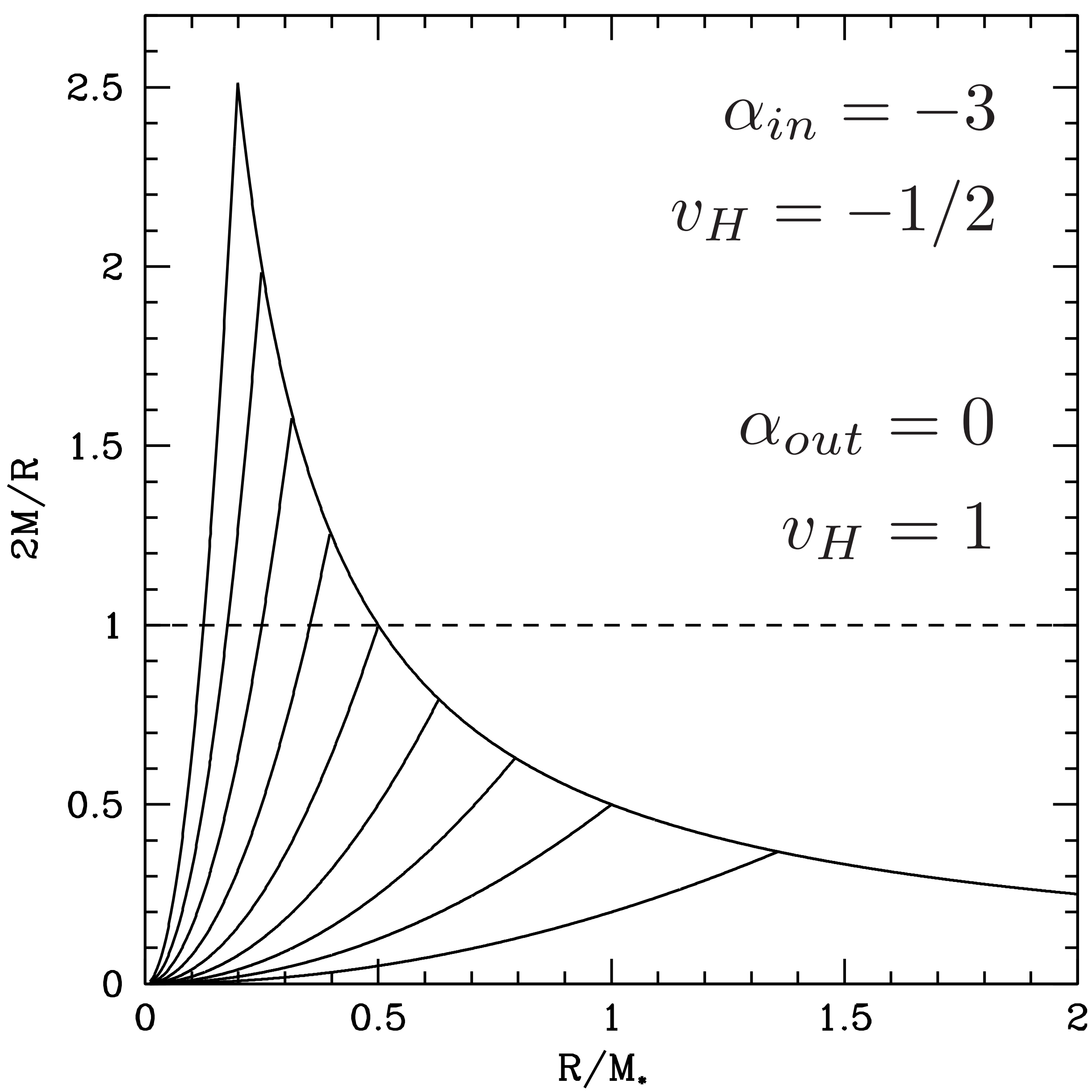
- Polytropic fluid: $p = K(s)\rho^\gamma$ $(\gamma = 5/3,\ 4/3,\ 2)$
    - If the fluid is adiabatic (no entropy change): $K(s) = K$ (constant)

# General Scheme for in/out-going horizon evolution



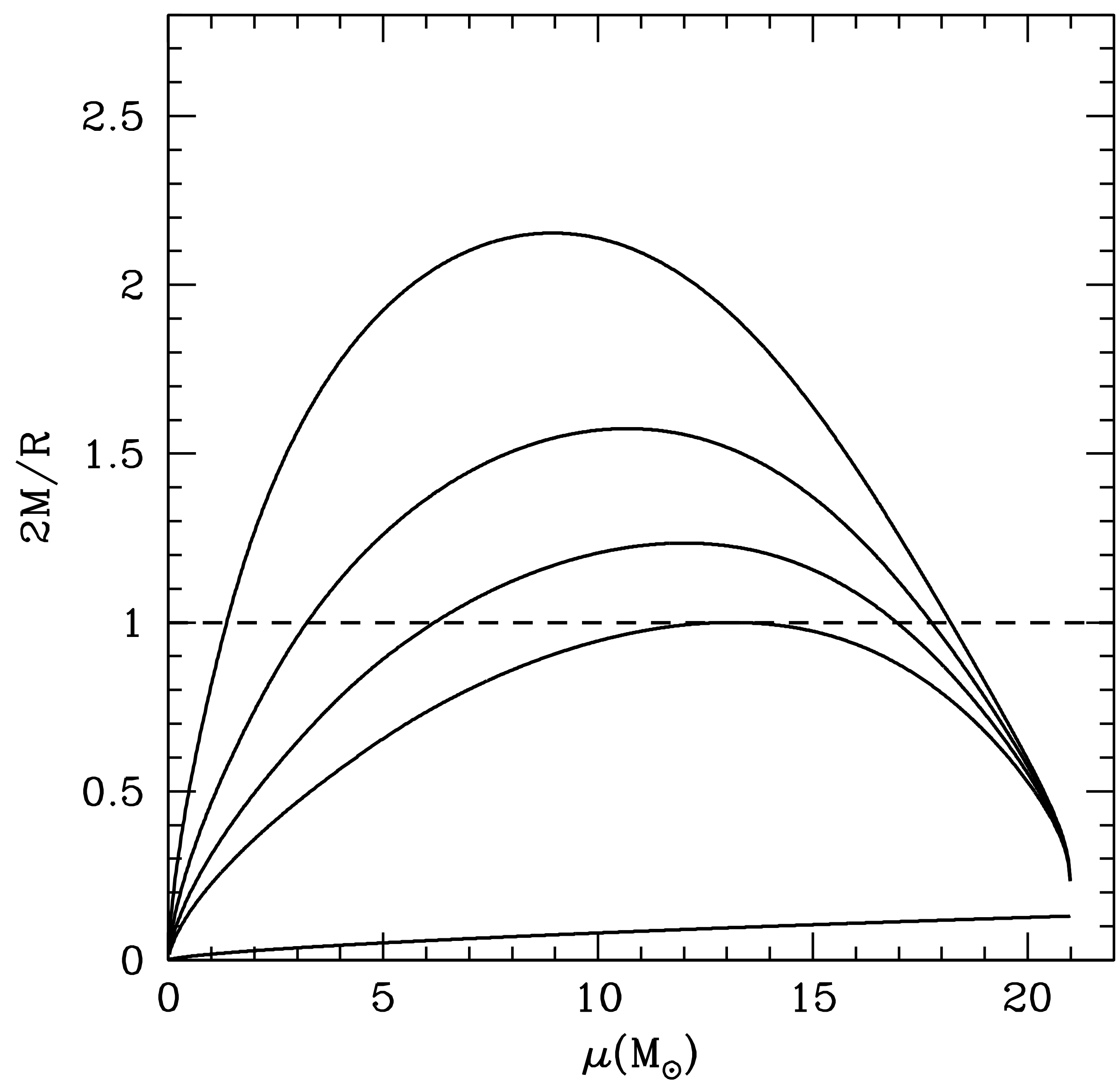
## *Oppenheimer-Snyder (1939)* homogenous collapse

$$\frac{2M}{R} = \frac{8}{3}\pi R^2 e \qquad p = 0$$




## *May & White (1966):* non homogenous collapse

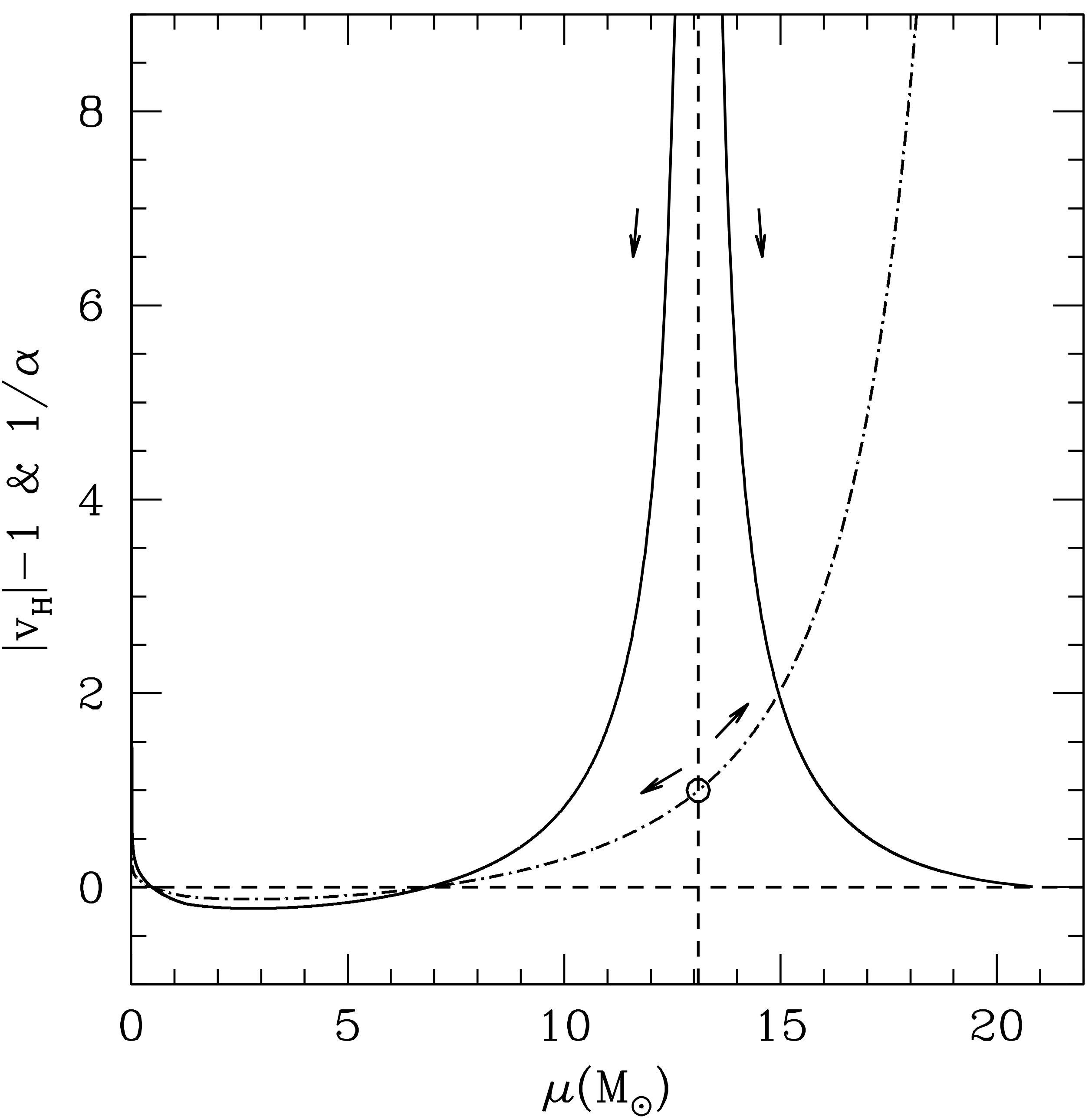
$$p = K\rho^\gamma \quad (\gamma = 5/3)$$

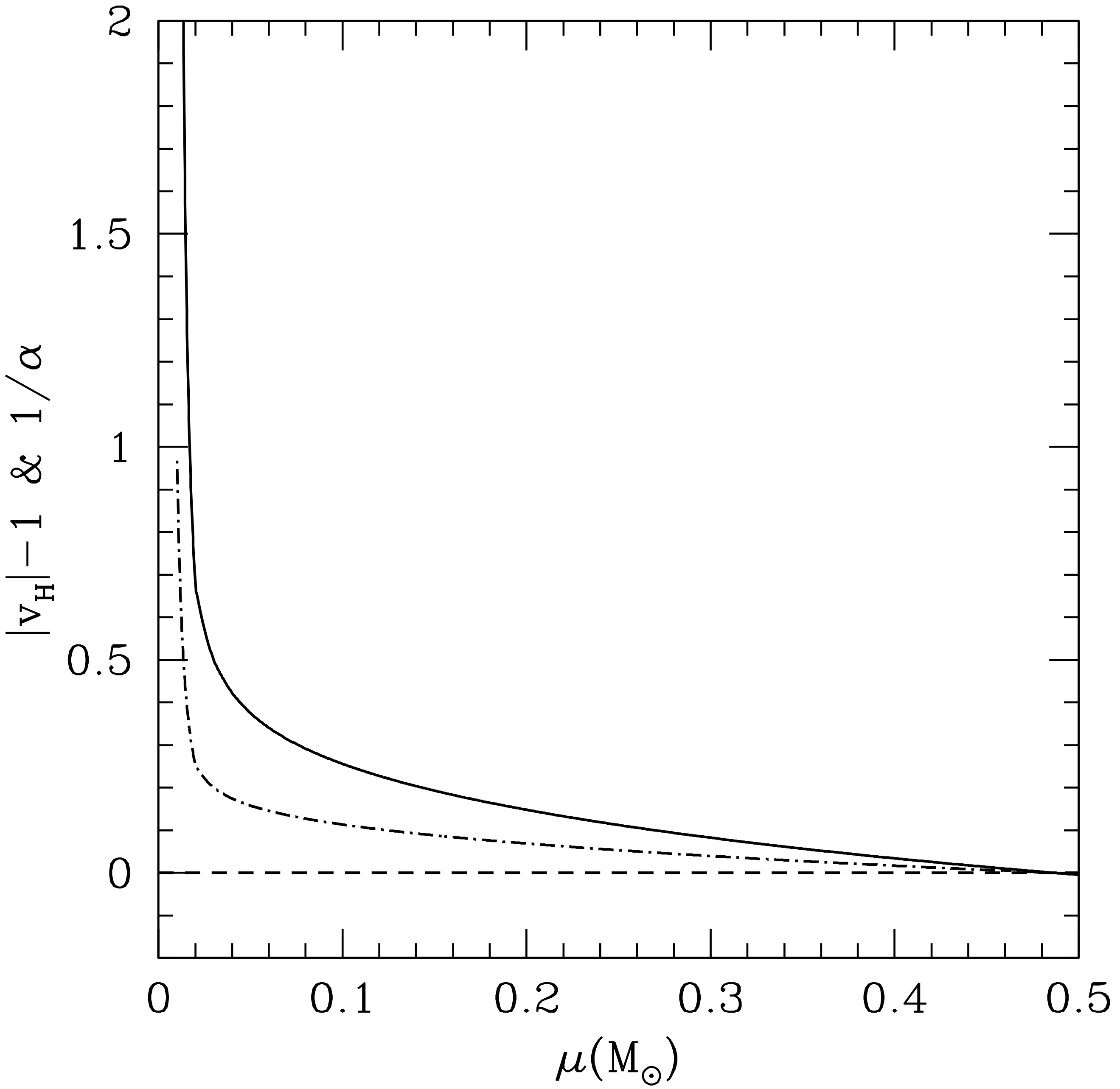


$$p = K\rho^{\gamma} \ (\gamma = 5/3, \text{HOM I.C.})$$

$$\alpha = \frac{4\pi R_H^2 (e+p)}{1 - 4\pi R_H^2 (e-p)}$$

$$v_H = \frac{1 + 8\pi R_H^2 p}{1 - 8\pi R_H^2 e}$$

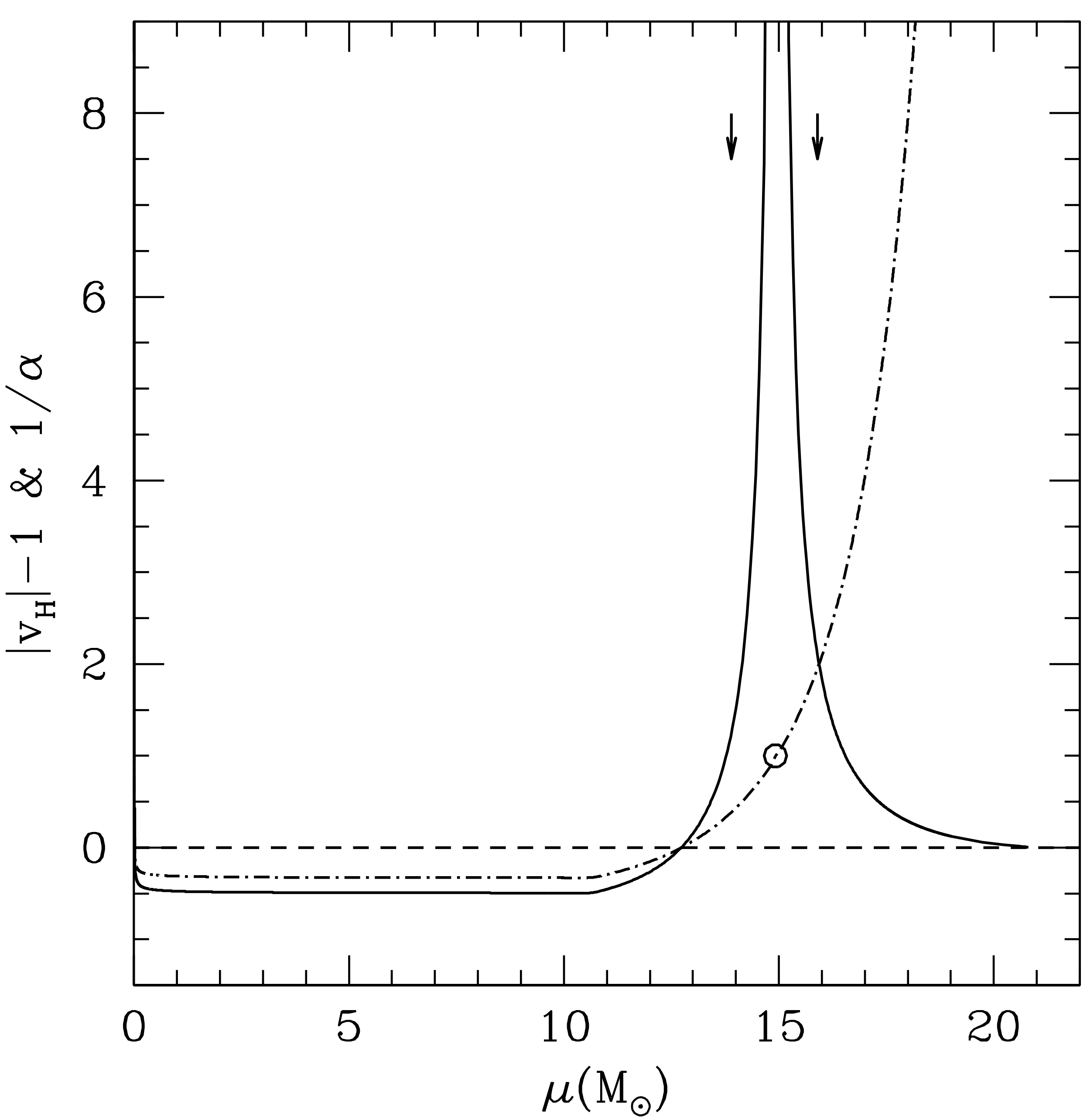


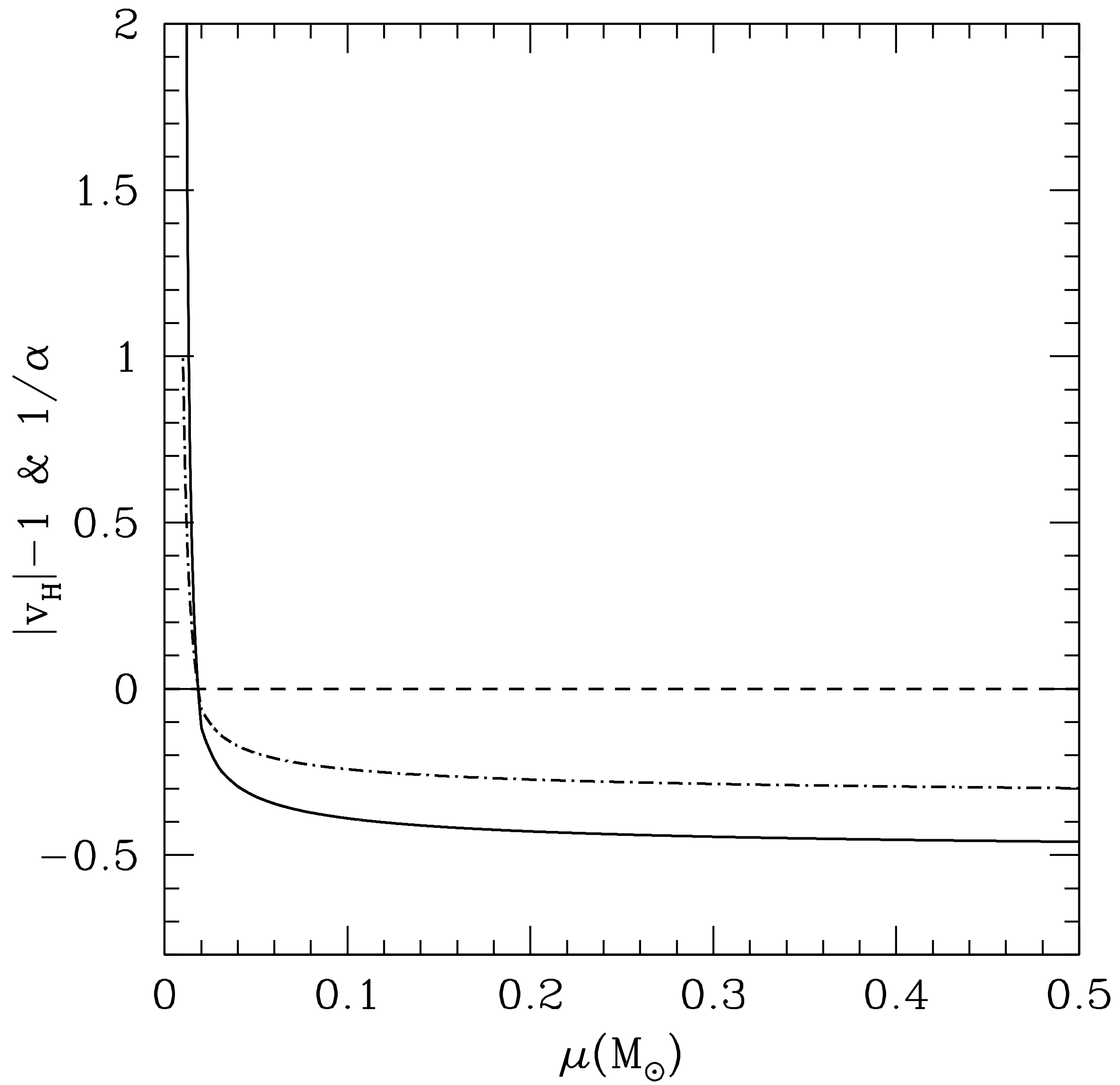



$$p = K\rho^{\gamma} \quad (\gamma = 4/3, \text{ HOM I.C.})$$

$$\alpha = \frac{4\pi R_H^2 (e+p)}{1 - 4\pi R_H^2 (e-p)}$$

$$v_H = \frac{1 + 8\pi R_H^2 p}{1 - 8\pi R_H^2 e}$$

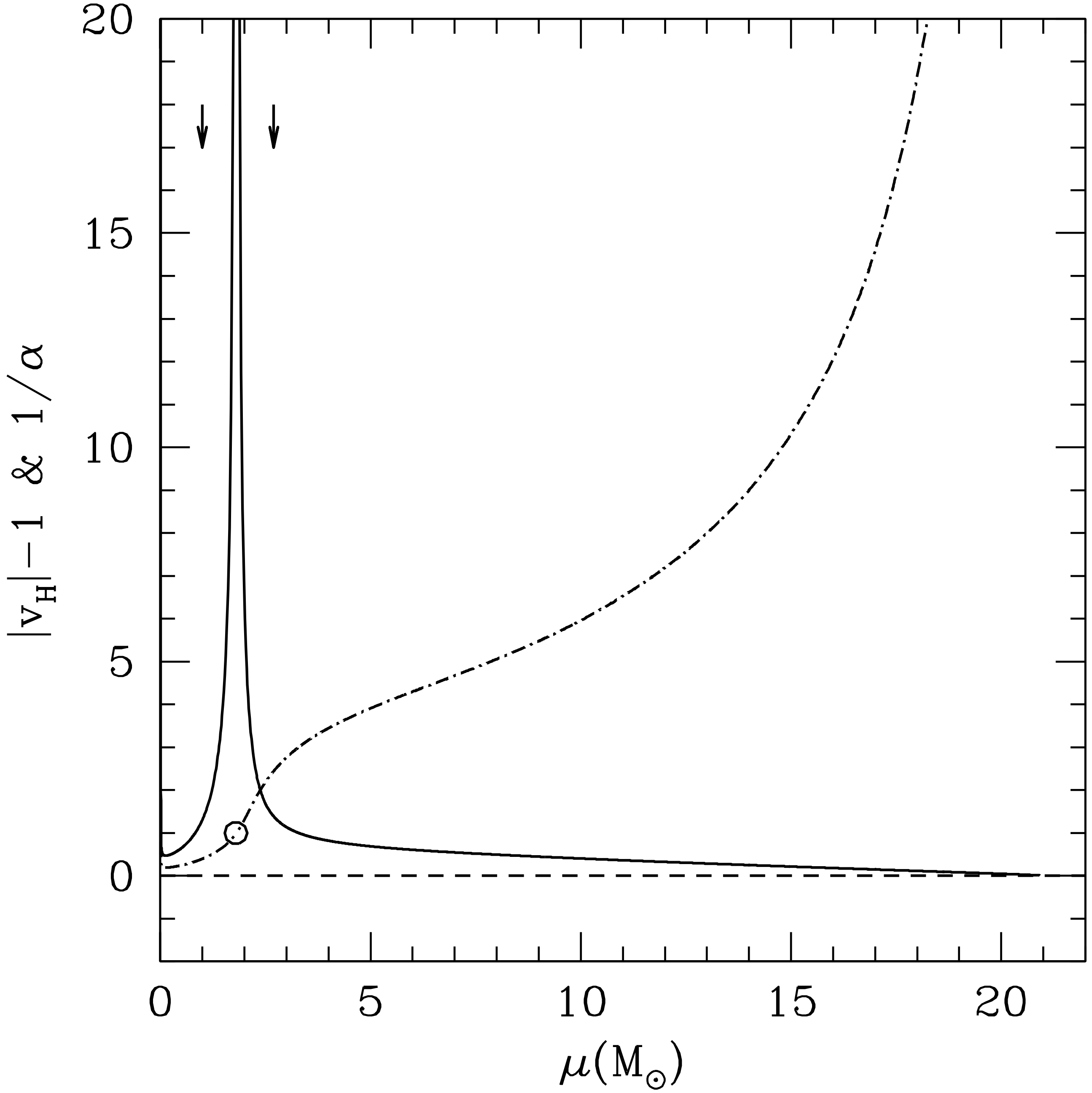


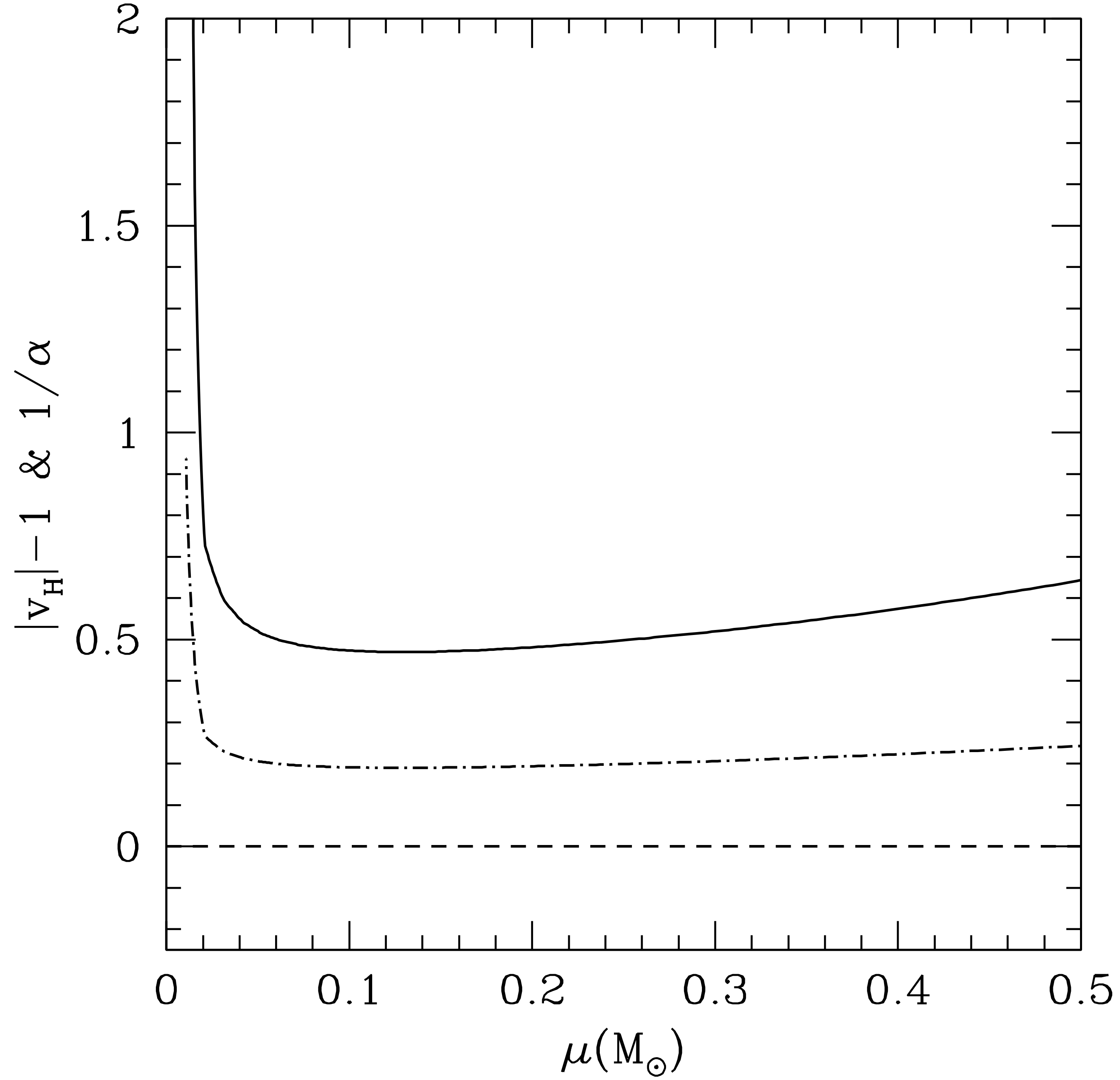



$$p = K\rho^{\gamma} \quad (\gamma = 5/3, \text{TOV I.C.})$$

$$\alpha = \frac{4\pi R_H^2(e+p)}{1 - 4\pi R_H^2(e-p)}$$

$$v_H = \frac{1 + 8\pi R_H^2 p}{1 - 8\pi R_H^2 e}$$



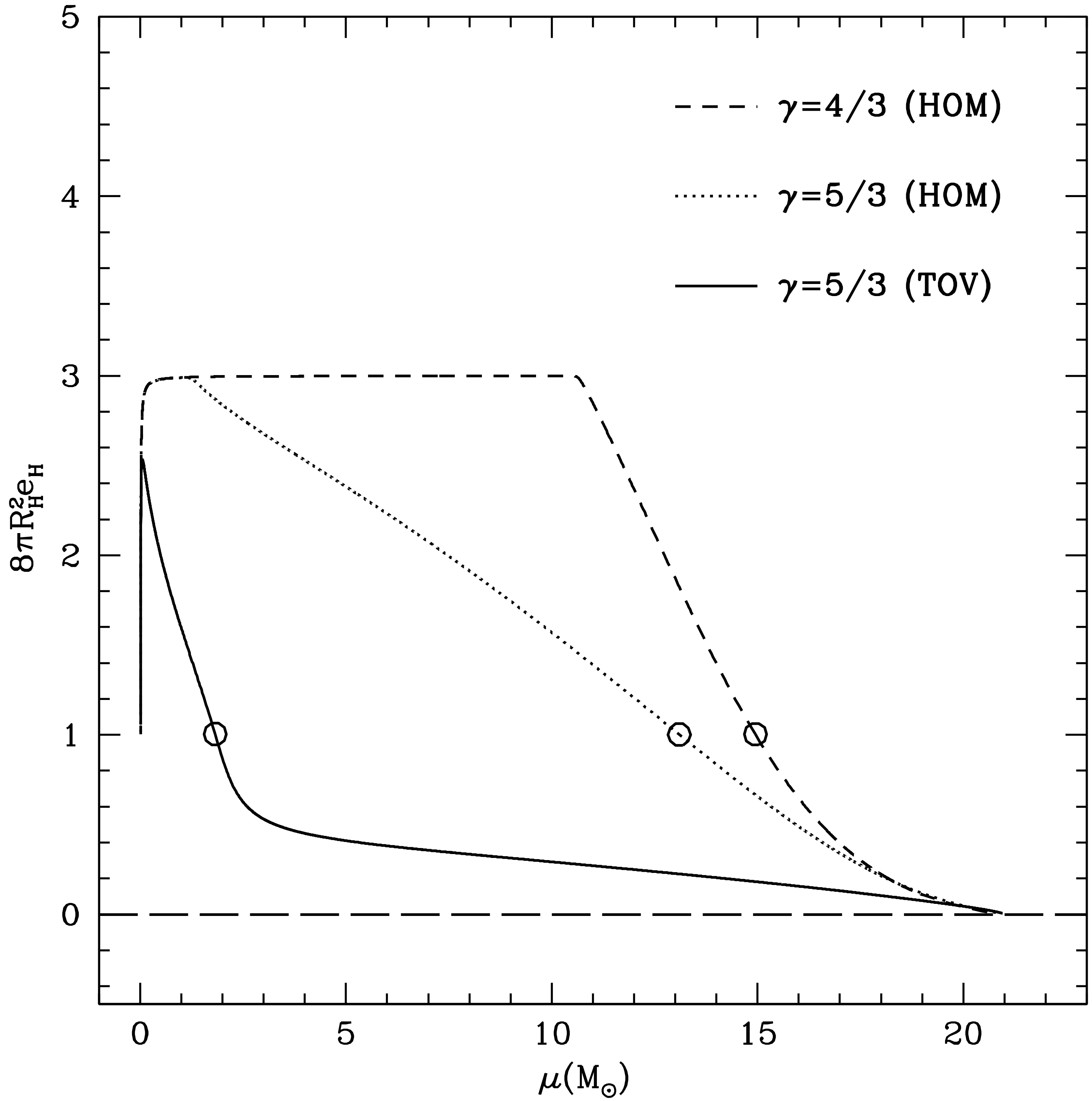



# Simulation Summary

$$\alpha = \frac{4\pi R_H^2(e+p)}{1-4\pi R_H^2(e-p)}$$

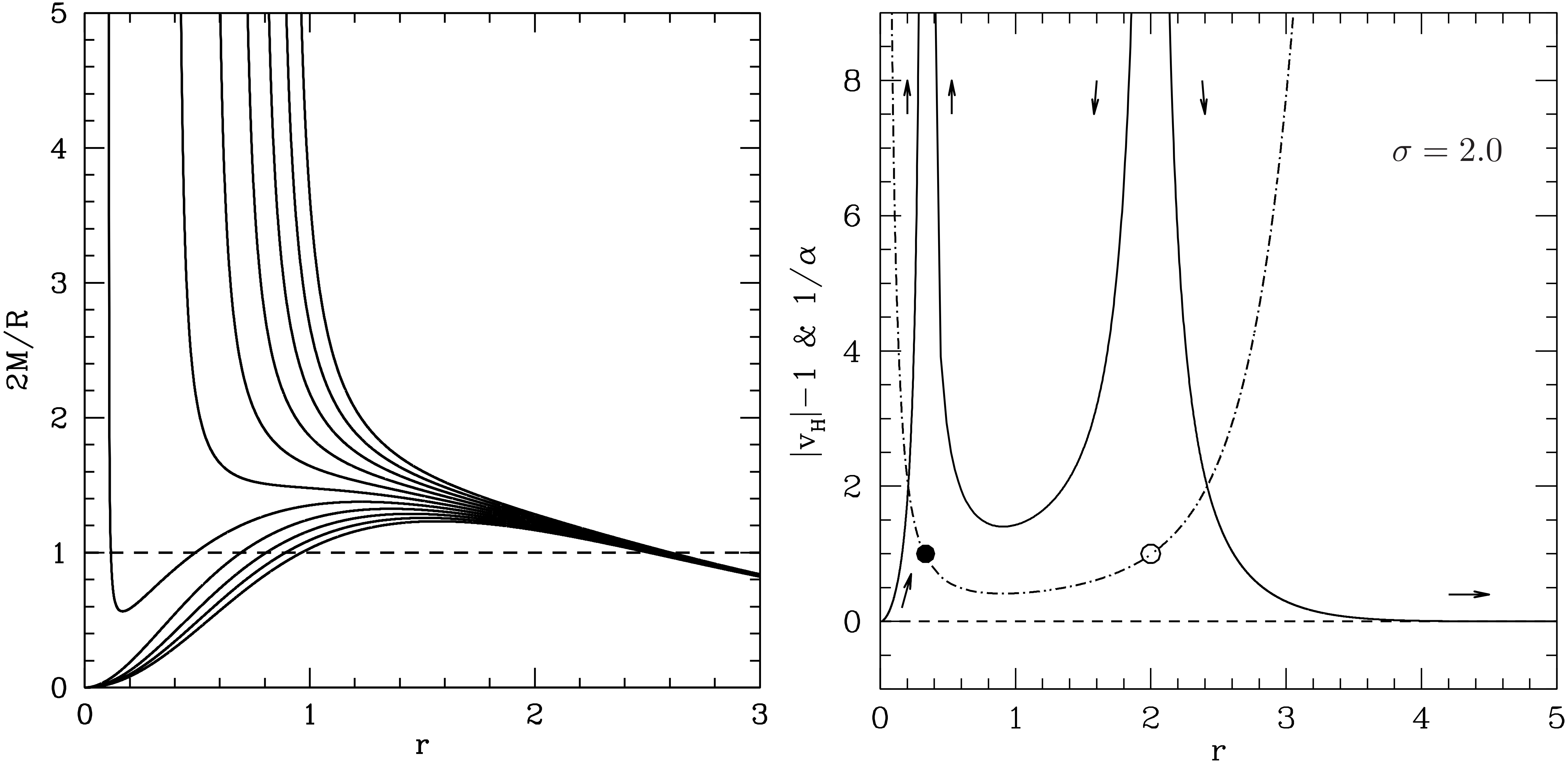
$$v_H = \frac{1+8\pi R_H^2 p}{1-8\pi R_H^2 e}$$

$$(\alpha = 1) \Rightarrow \; e = \frac{1}{2A_H}$$



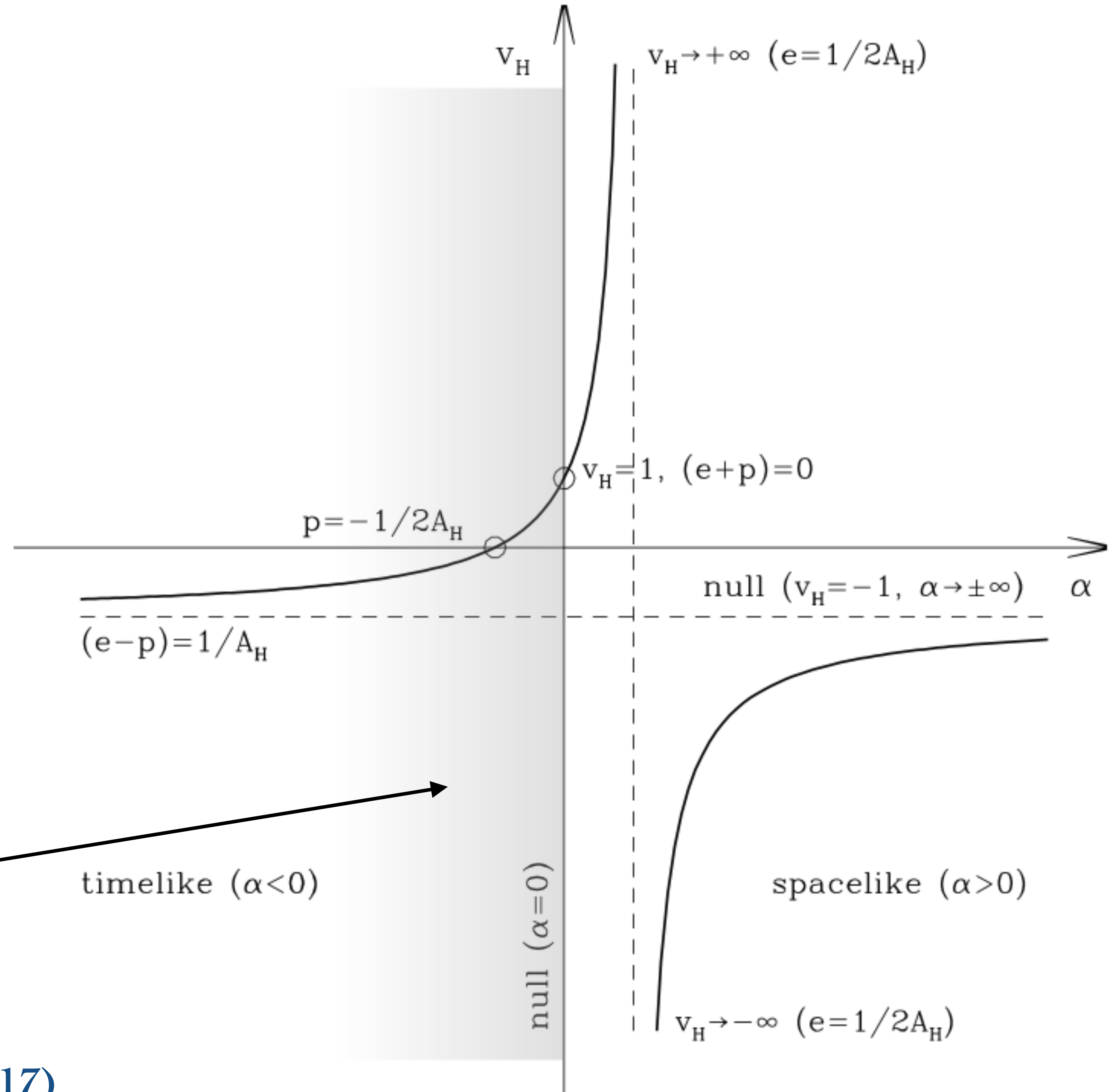
# LTB collapse

**THREE HORIZONS:** If the ingoing horizon is not reaching the center when the singularity is forming, a second outgoing horizon is originated from the singularity which is going to annihilate with the ingoing horizon at $\alpha = 1$ and $v_H = \infty$ .



# Black Hole Horizons - Phase Diagram

$$v_H = -\frac{\alpha + 1}{\alpha - 1}$$




**Negative Pressure**

A. Helou, I.M., J. Miller - CQG (2017)

V. Faraoni, G. Ellis, J. Firouzjaee, A. Helou, I.M. - PRD (2017)

# Conclusions & Future perpectives

- With the Misner-Sharp equation (cosmic time slicing) we have studied the **causal nature of trapping horizons** appearing in **gravitational collapse** forming black holes.
- Within the classical regime of GR we have observed space-like outgoing horizons and space-like/time-like ingoing horizons (equation of state and initial conditions for density).
- **Pressure plays a key role - Cosmic Censorship.**
- The conditions of horizon formation and annihilation are independent of the initial conditions.
- The formalism developed to show the possibility of incorporating quantum effects within the classical formulation of the GR-hydro equations modifying the equation of state accordingly to quantum gravity. Is it possible to obtain **non singular BH?**

  ***C. Bambi, D. Malafarina & L. Modesto (2013)***

  ***C. Rovelli & F. Vidotto (2014);***

  ***A. Helou, D. Malafarina & I.M. - in progress***

- The formalism can be also to the cosmological horizon, studying causal nature evolution for a non homogenous Universe.

  ***I.M, A. Helou - in progress***