Dirac tunneling: superluminal velocities and closed time-like curves?

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The Hartman Effect

In 1932, **L. A. MacColl** reported that when wave packets tunnel through a rectangular potential barrier, in one dimension, they appear on the other side of the barrier at the same time they first impinge on the barrier.

In 1962, Thomas Hartman made similar observations –

This is known as the **Hartman effect**.

Two observations:

1. The portion of the wave packet that tunnels through the barrier travels faster.
2. The tunneled particle appears to spend **no time** under the barrier.

Arbitrarily wide barrier → **arbitrarily large effective velocity**.

Superluminal Effective Velocities have been reported

e.g., S. De Leo & P.P. Rotelli 2007, using stationary phase approximation of Dirac wavepacket propagation.

Many questions remain: Effective Velocity = **Distance** / **Time**

Many means of characterizing **time** have been proposed.

One generally considers the narrow-in-energy wavepacket limit, and defines a mean time related to the derivative of the phase or log-modulus of the transmission amplitude with respect to barrier height or particle energy.

We are interested in narrow-in-space wavepackets, and associated **arrival time distributions**:

Consider a barrier extending from *a* to *b*, and a particle incoming from the left. It is initially localized about z_0.

Probability that the particle is not beyond the barrier at time = *t* :

$$C(t) = \int_{-\infty}^{b} dz \, \left|\psi_t(z)\right|^2 = \left\langle \psi_t \middle| \hat{\theta}(-\infty, b)\psi_t \right\rangle = \left\langle \psi \middle| \hat{\theta}_{-t}(-\infty, b)\psi \right\rangle$$

$$\hat{\theta}_{-t}(-\infty, b) = \exp\left(i\widehat{H}t/\hbar\right)\hat{\theta}(-\infty, b)\exp\left(-i\widehat{H}t/\hbar\right)$$

$$\frac{d}{dt}\hat{\theta}_{-t}(-\infty, b) = \frac{i}{\hbar}\left[\widehat{H}, \hat{\theta}_{-t}\right] = \hat{J}_{-t}$$

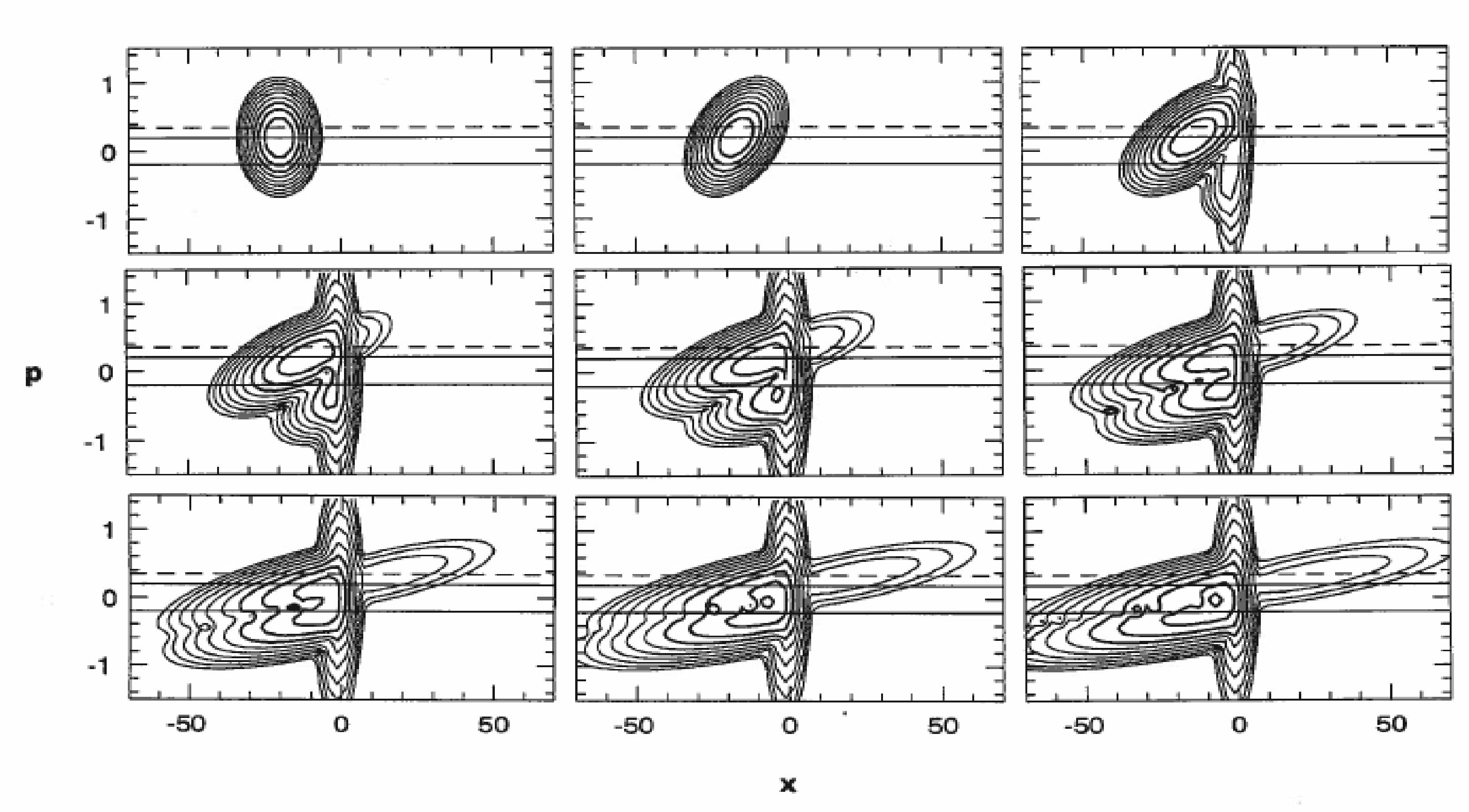
Probability distribution (normalized to total transmission probability) for **arrival** at *b*:

$$P(t) = -\frac{dC(t)}{dt} = \left\langle \psi \middle| \hat{J}_{-t}(b)\psi \right\rangle$$

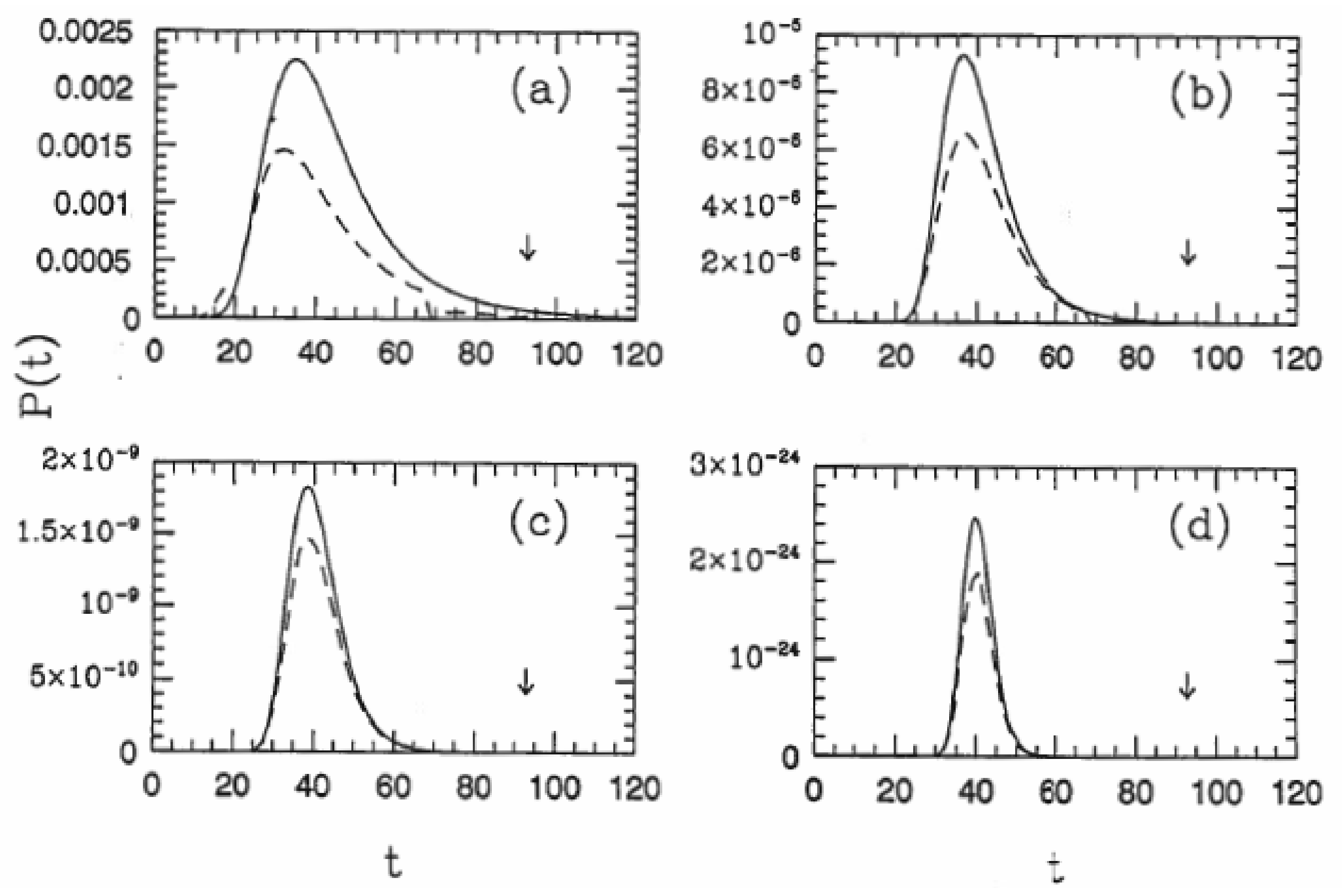
$$= \frac{\hbar}{m}\mathrm{Im}\left[\psi_t(b)\frac{d}{dx}\psi_t(b)\right] \qquad \text{non-relativistic case}$$

Schrödinger Tunneling through an Eckart Barrier

Husimi transforms of time evolving wavepacket – initially, a gaussian

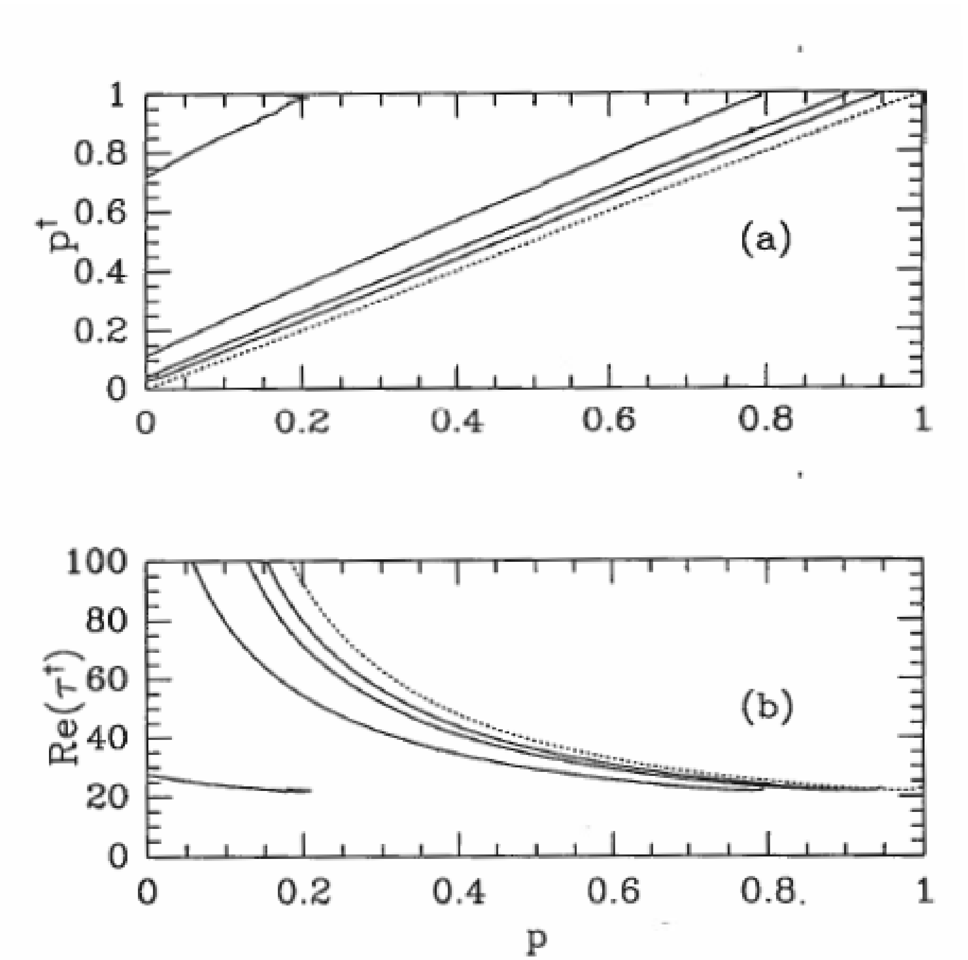


Transmitted fluxes



Arrows show arrival time of center of initial gaussian with no barrier.
Dashed lines are semiclassical approximations.

Stationary phase approximation:



$$\psi(z,t)$$

$$= \int_0^\infty dp' \, c(p')\psi_{p'}(z)\exp\left(-\frac{iE(p')t}{\hbar}\right)$$

$$= \int_0^\infty dp' \exp\left(-\frac{(p'-p)^2}{2\delta p^2} + \frac{iW(E',z)}{\hbar} - \frac{ip'^2 t}{2m\hbar}\right)$$

$$-\frac{p^\dagger - p}{\delta p^2} + \frac{i}{\hbar}\left(\frac{dE}{dp}\frac{dW}{dE}\right)^\dagger - \frac{ip^\dagger t}{m\hbar} = 0$$

$$\frac{p^\dagger - p}{p^\dagger} = i\frac{\tau(p^\dagger) - t}{\tau_0}$$

$$\tau_0 = \frac{m\hbar}{\delta p^2} = \frac{\delta x}{\delta p / m}$$

Most probable time:

$$t = \mathrm{Re}\,\tau(p^\dagger) \quad \text{and}$$

$$\frac{p^\dagger - p}{p^\dagger} = -\frac{\mathrm{Im}\,\tau(p^\dagger)}{\tau_0}$$

Relativistic Quantum Mechanics – the **Dirac Equation**

Special Relativity:

$$E - V = \sqrt{m^2c^4 + c^2p^2}$$
$$= \gamma mc^2$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Goal: Quantize the above (classical) equation

The **Schrödinger Equation** results from quantizing $E - V = \dfrac{p^2}{2m}$

$$\left(i\hbar\frac{\partial}{\partial t} - V\right)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi$$

The **Klein-Gordon Equation** results from quantizing $E^2 = m^2c^4 + c^2p^2$

$$-\hbar^2 \frac{\partial^2}{\partial t^2}\psi = \left(m^2c^4 - c^2\hbar^2\nabla^2\right)\psi$$

Dirac wanted a first order equation in time:

$\psi(z,t)\big|_{t=0}$ is the state of the system at time = 0
uniquely determines the state at all other times.

The Dirac Equation (in 3+1 dimensions):

$$\gamma_0 i\hbar \frac{\partial}{\partial t}\psi = \left(mc^2 + c\vec{\gamma}\bullet\left(-i\hbar\vec{\nabla}\right)\right)\psi$$

Add potential →

$$\gamma_0\left(i\hbar\frac{\partial}{\partial t} - V\right)\psi = \left(mc^2 + c\vec{\gamma}\bullet\left(-i\hbar\vec{\nabla}\right)\right)\psi$$

ψ is a 4-component vector

If $V = V(z)$ the above reduces to the 1+1 dimensional Dirac equation.

ψ is now a 2-component vector - spin is conserved

$$\tilde{\gamma}_0\left(i\hbar\frac{\partial}{\partial t}-V\right)\psi=\left(mc^2+c\tilde{\gamma}_3\left(-i\hbar\frac{\partial}{\partial z}\right)\right)\psi$$

$$\tilde{\gamma}_0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad\text{and}\quad\tilde{\gamma}_3=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

The Time Independent Dirac Equation - 1+1 dimensional

$$\tilde{\gamma}_0\left(E-V\left(z\right)\right)\psi=\left(mc^2+c\tilde{\gamma}_3\left(-i\hbar\frac{\partial}{\partial z}\right)\right)\psi$$

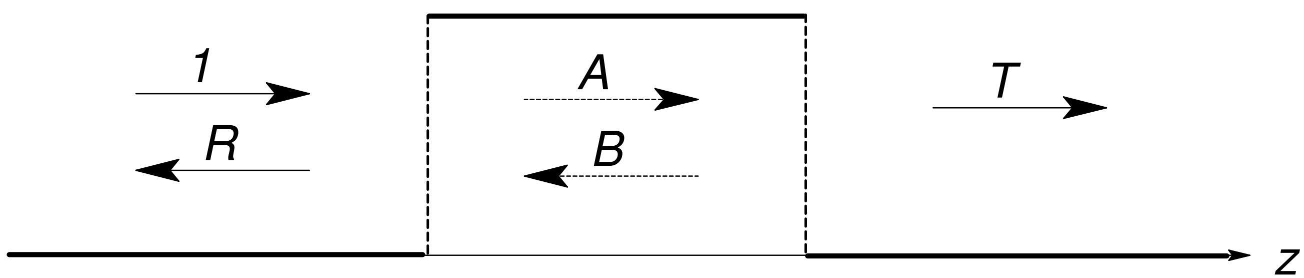
$$\widehat{H}=mc^2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}-i\hbar c\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{\partial}{\partial z}+V\left(z\right)$$

Positive energy solutions (constant *V*):

$$\psi_p(z) = \begin{pmatrix} 1 \\ \dfrac{\pm cp}{E - V + mc^2} \end{pmatrix} \exp\left(\pm \frac{ipz}{\hbar} \right)$$

$$p = \sqrt{\frac{(E-V)^2}{c^2} - m^2c^2}$$

Solutions for piece-wise constant potential are obtained by imposing **continuity** of ψ at the boundaries – **transfer matrix solution**



Dirac Wavepacket Dynamics

$$\psi(z,t)=\int_0^\infty dp\, c(p)\psi_p(z)\exp\left(-\frac{iE(p)t}{\hbar}\right)$$

Dirac Flux and arrival time distribution:

$$P(t)=-\frac{dC(t)}{dt}=\left\langle\psi\middle|\hat{J}_{-t}(b)\psi\right\rangle$$
$$=c\,\mathrm{Im}\left[\psi_t(b)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\psi_t(b)\right]$$

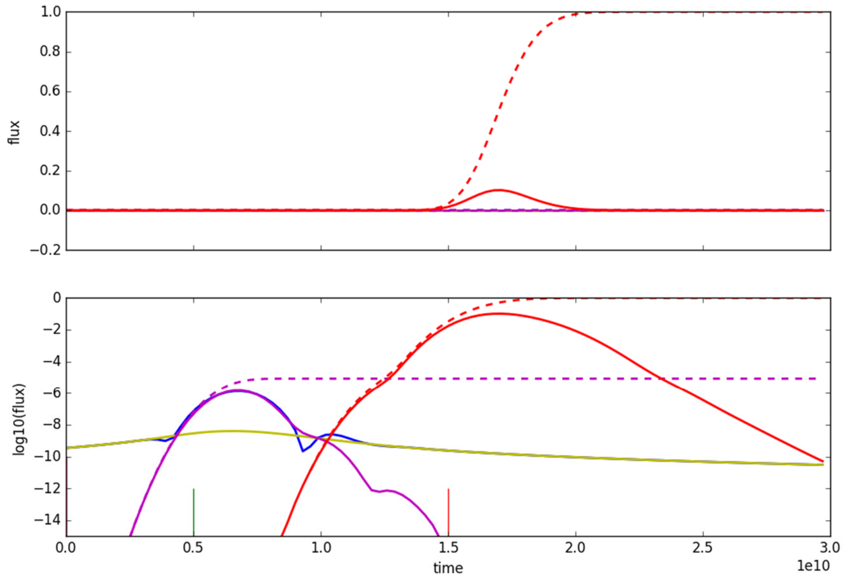
Semiclassical approximation (stationary phase):

$$\frac{p^\dagger-p_0}{p^\dagger}=i\frac{\tau(p^\dagger)-t}{\gamma^\dagger\tau_0} \quad \text{or} \quad p^\dagger-p_0=iv^\dagger\frac{\tau(p^\dagger)-t}{\tau_0}$$

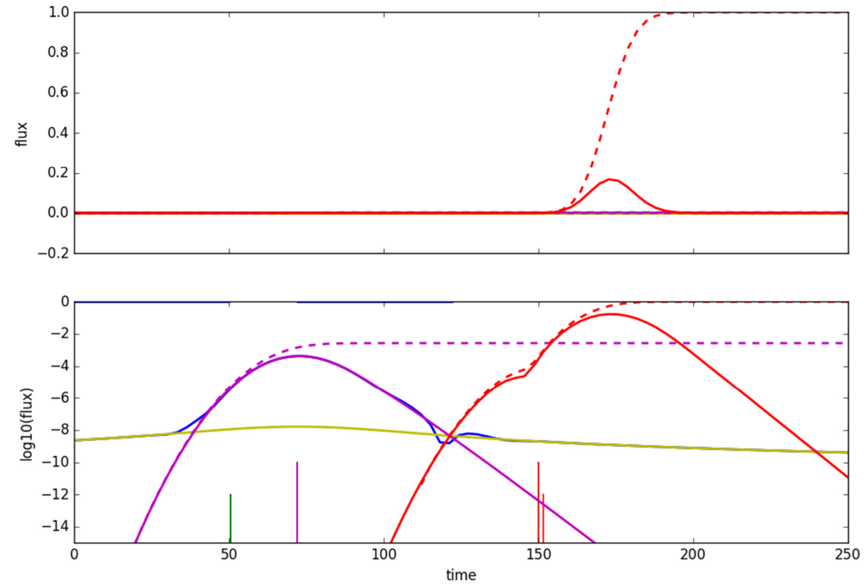
Non-relativistic and Relativistic Arrival time distributions

distance unit = reduced Compton wavelength = ƛ = 3.86 x 10^{-13} m
time unit = ƛ / c = 1.29 x 10^{-21} s = 1.29 zs
energy unit = electron rest mass energy = 0.511 MeV

$v = 10^{-4}$ $\delta x = 10^5$ $V_{top} = 10^{-8}$



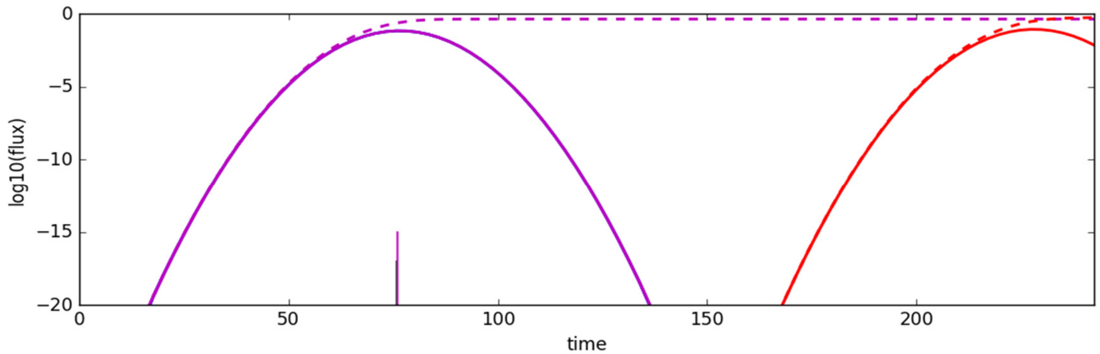
$v = 0.99$ $\delta x = 10$ $V_{top} = 7$



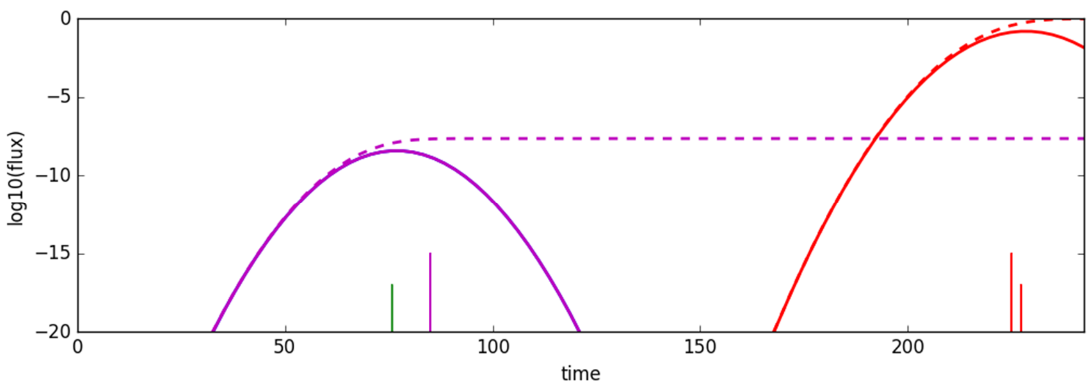
Potential is a staircase with ten steps up, then 10 steps down.
Step width = 10^4 and 1, respectively.

Rectangular Barrier $\delta x = 9$

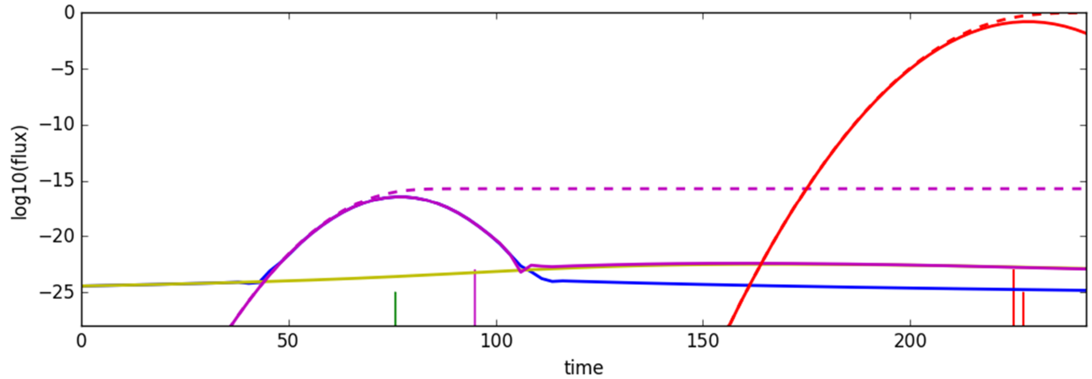
$v = 0.99 \quad w = 1 \quad V_{top} = 6.8$



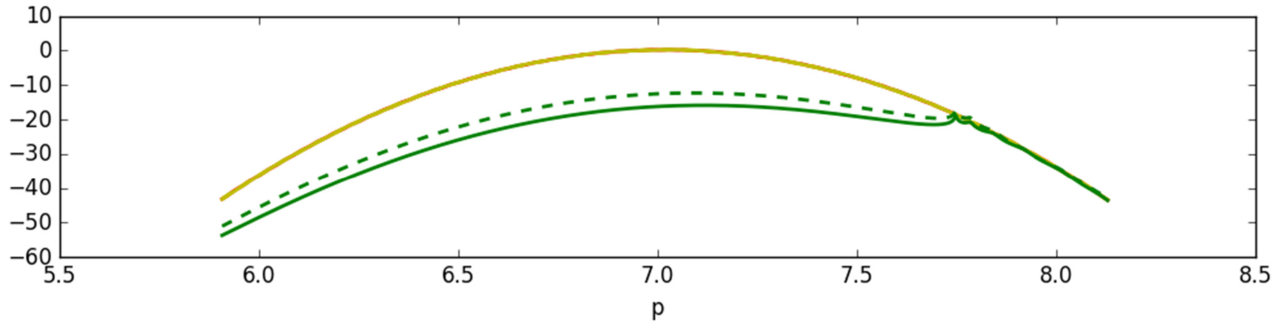
$v = 0.99 \quad w = 10 \quad V_{top} = 6.8$



$v = 0.99 \quad w = 20 \quad V_{top} = 6.8$

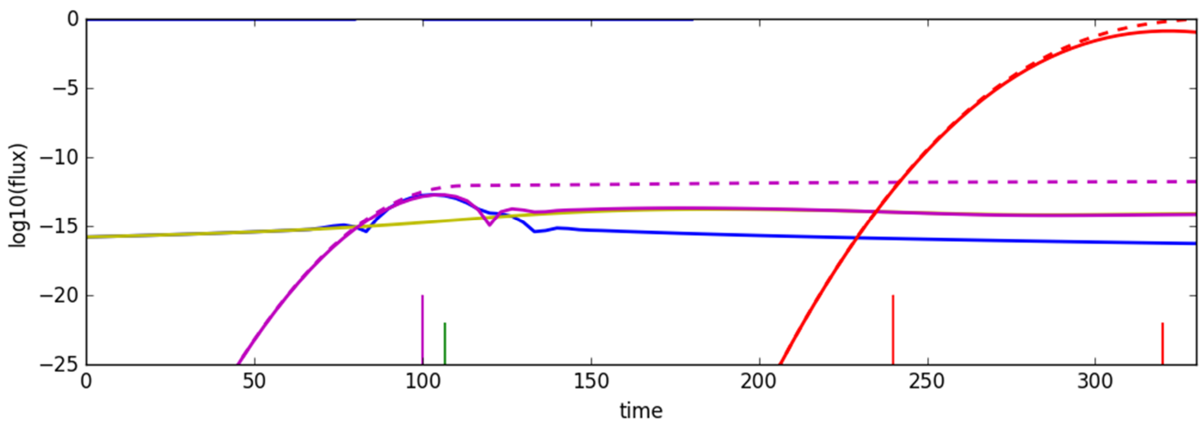


|Integrand| for three z values

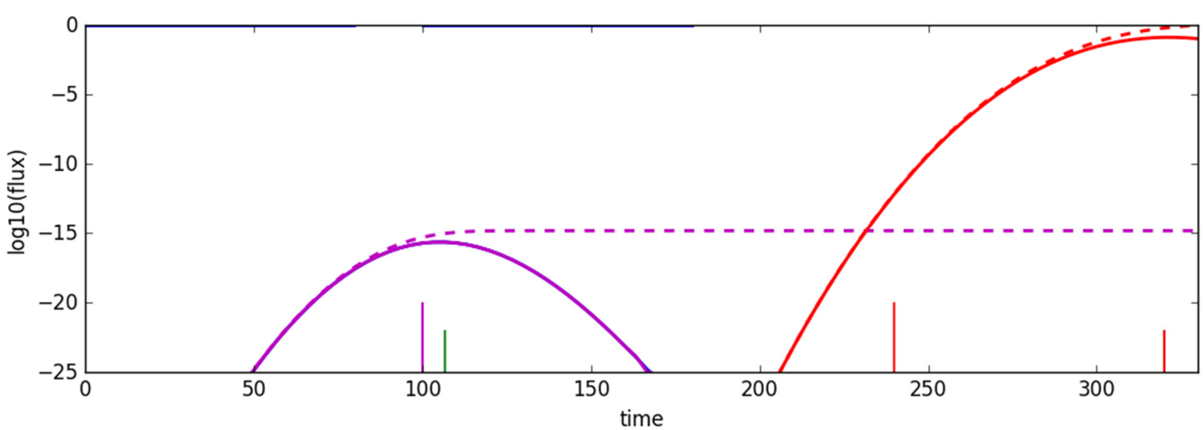


Rectangular Barrier with $w = 20$

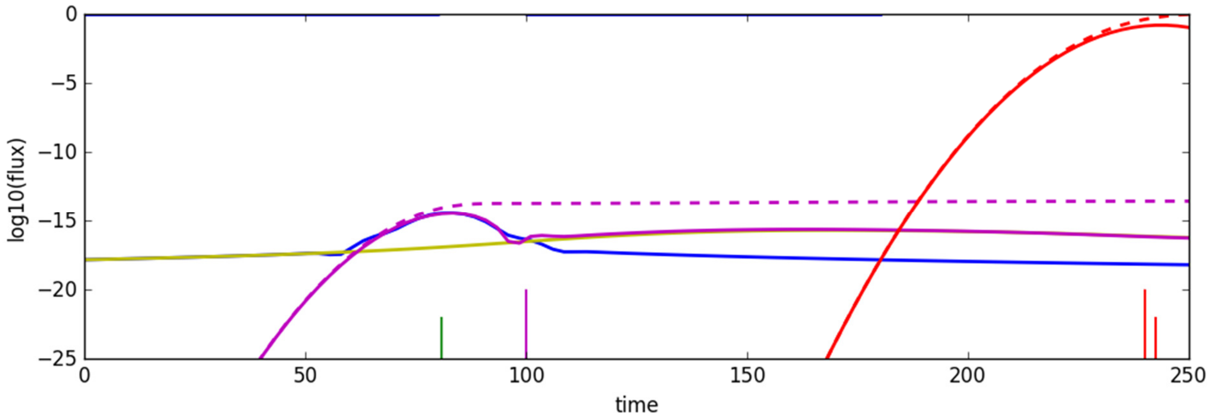
$v = 0.75 \quad \delta x = 10 \quad V_{top} = 0.9$



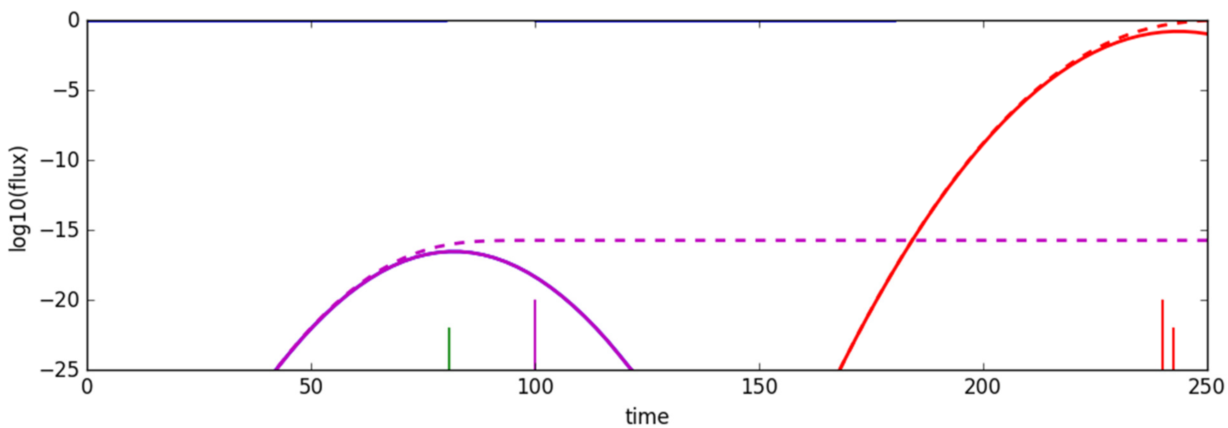
$V_{top} = 1.1$



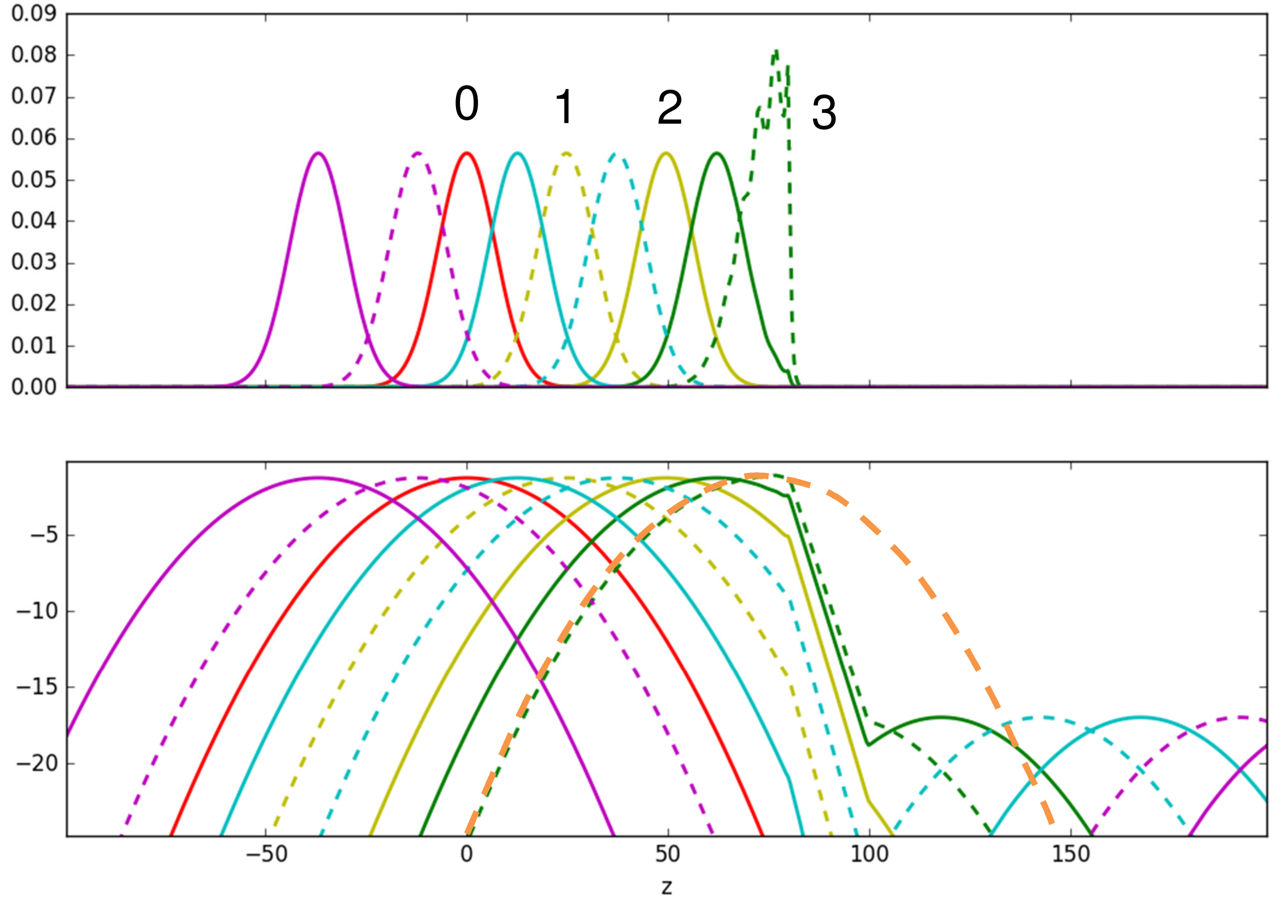
$v = 0.99 \quad \delta x = 10 \quad V_{top} = 6.6$



$V_{top} = 6.8$



Wavepacket Dynamics

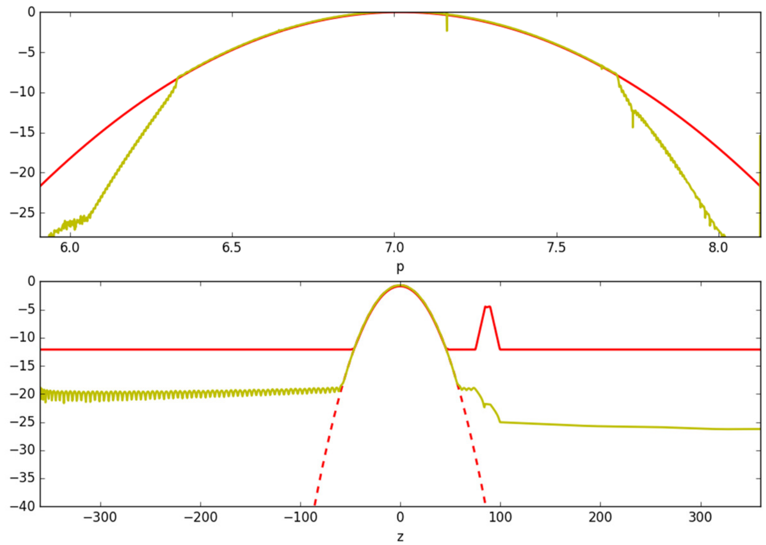


The Generalized Hartman Effect

It has been reported that, in the opaque barrier limit, the phase time is independent of the spacing between successive barriers.
V. S. Olkhovsky, E. Recami and G. Salesi, 2002.

Series of Barriers

$v = 0.99$ $\delta x = 9$ $w = 10, 5, 10$ $V_{top} = 6.8$



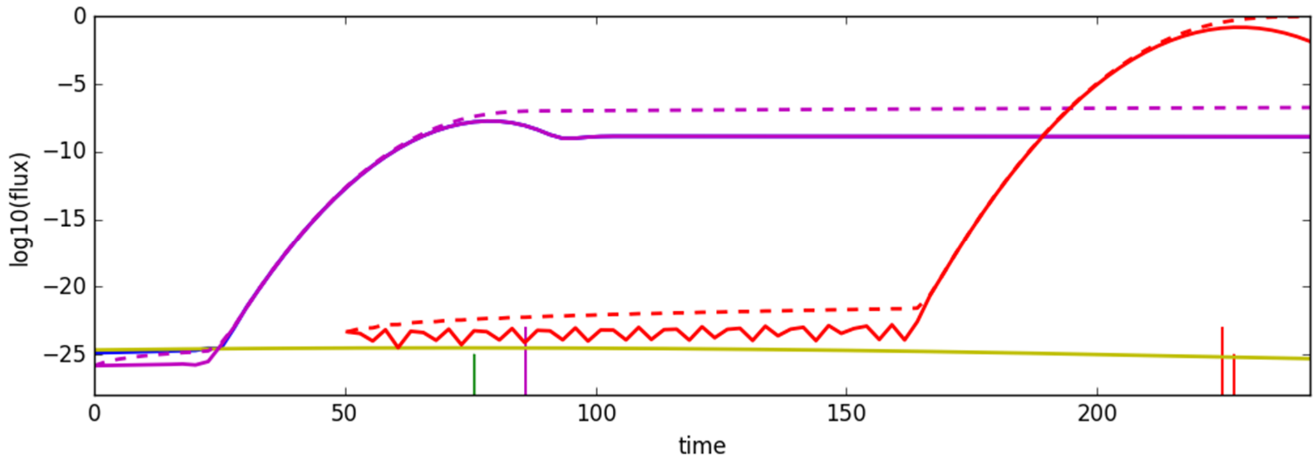
We can no longer use a simple Gaussian in *p* space (*p* is just a proxy for energy). The energy eigenfunctions, summed with Gaussian weighting in *p*, do not give a Gaussian in *z* space.

We use **singular value decomposition** to invert

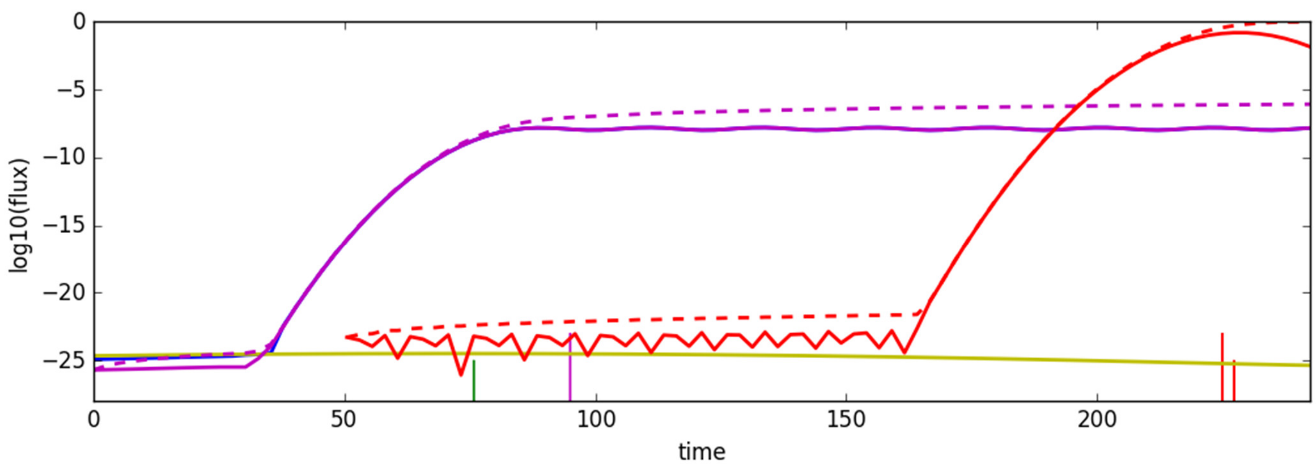
$$\psi(z,0) = \int_0^\infty dp\; r(p)\, c_0(p)\, \psi_p(z)$$

$$r(p) = \frac{c(p)}{c_0(p)}$$

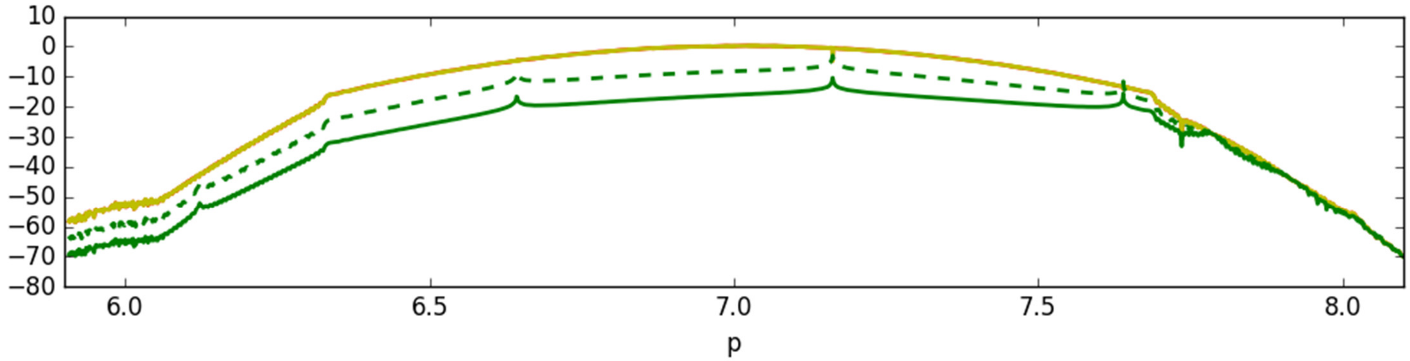
$v = 0.99 \quad \delta x = 9 \quad w = 5, 1, 5 \quad V_{top} = 6.8$



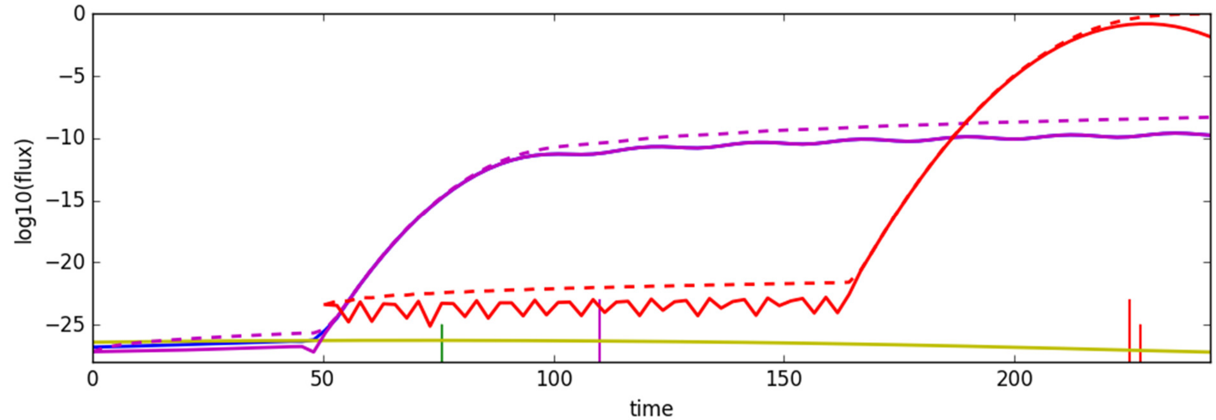
$v = 0.99 \quad \delta x = 9 \quad w = 5, 10, 5 \quad V_{top} = 6.8$



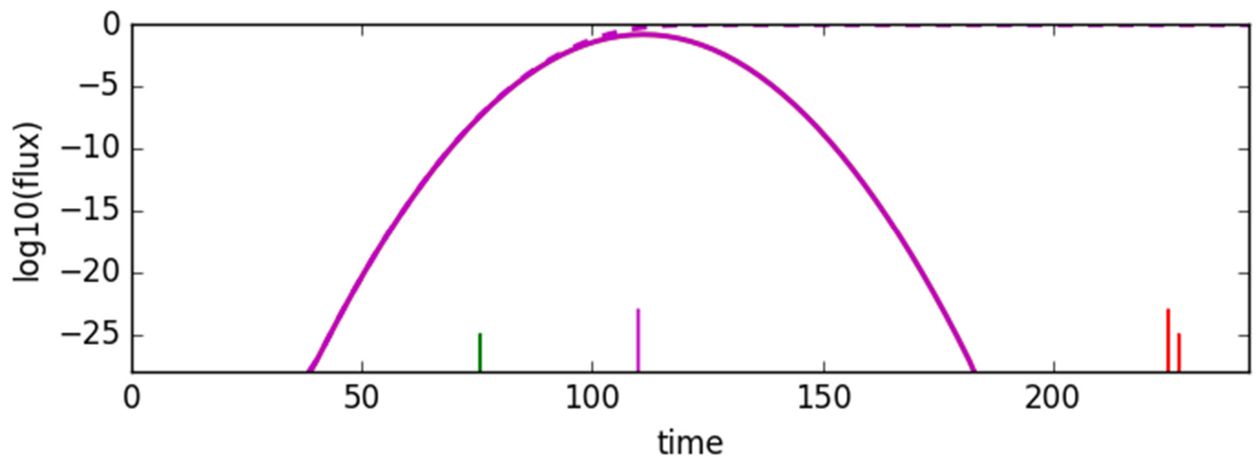
Integrand| for three z values



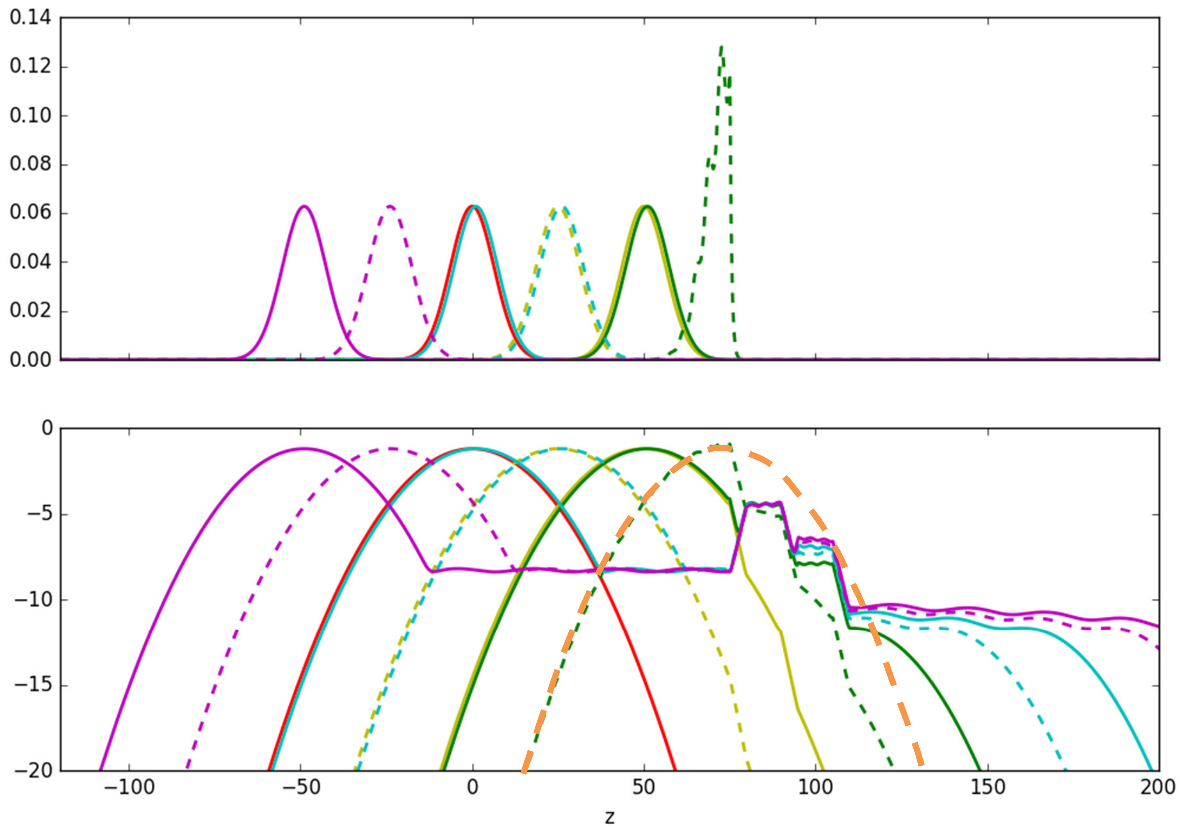
$v = 0.99 \quad \delta x = 9 \quad w = 5, 10, 5, 10, 5 \quad V_{top} = 6.8$



free particle $v = 0.99 \quad \delta x = 9 \quad V_{top} = 0$



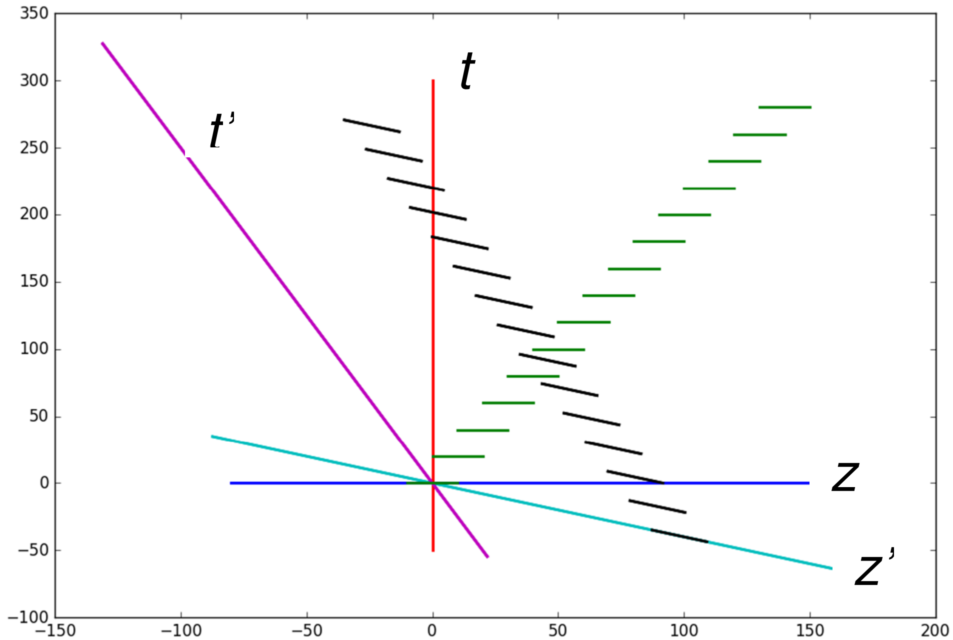
$v = 0.99 \quad \delta x = 9 \quad w = 5, 10, 5, 10, 5 \quad V_{top} = 6.8$


0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
−5
−10
−15
−20
−100
−50
0
50
100
150
200
z

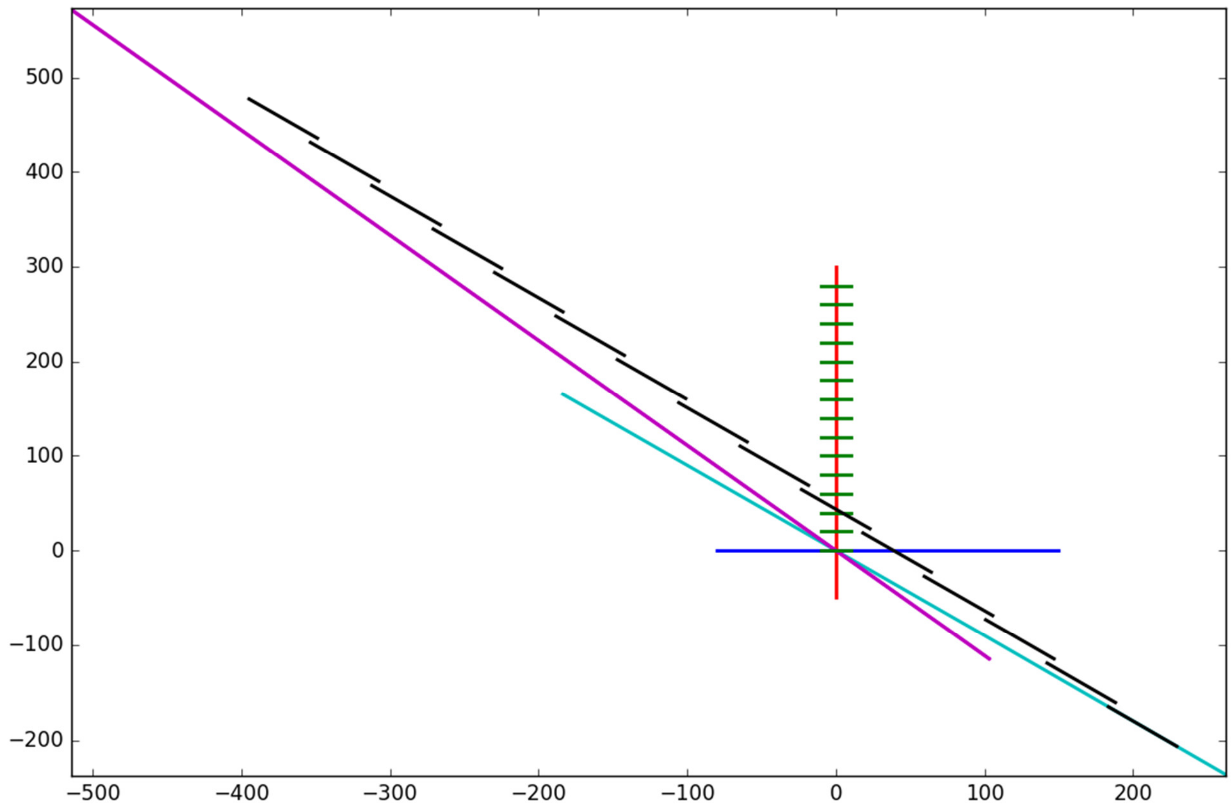
Moving Barrier

Lorentz transformation relates space and time coordinates in the observer and barrier frames of reference.

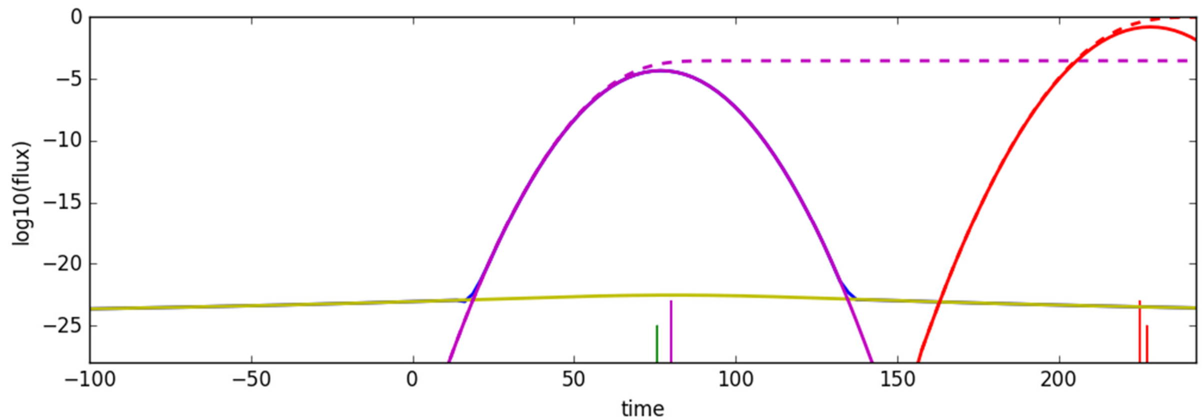
$v_e = 0.5$ and $v_{barrier} = -0.4$



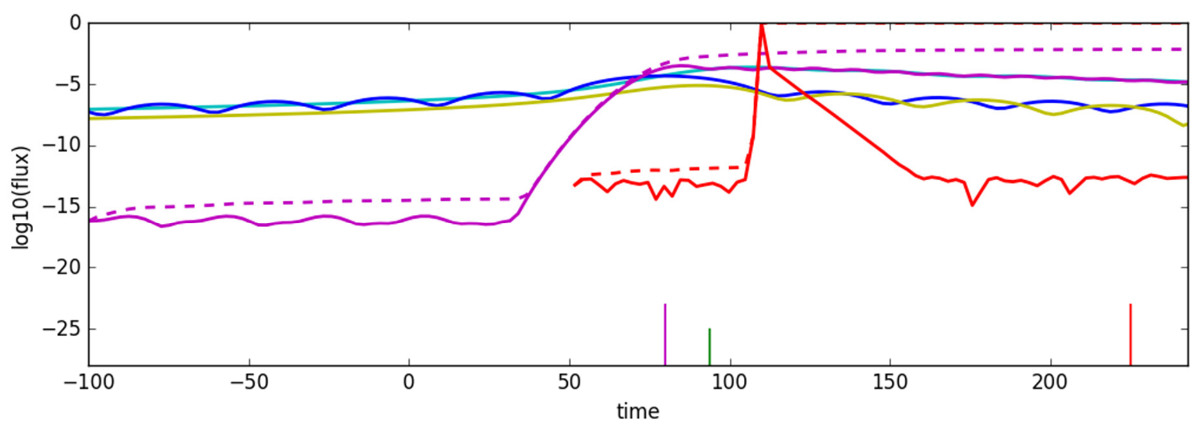
$v_e = 0$ and $v_{barrier} = -0.9$



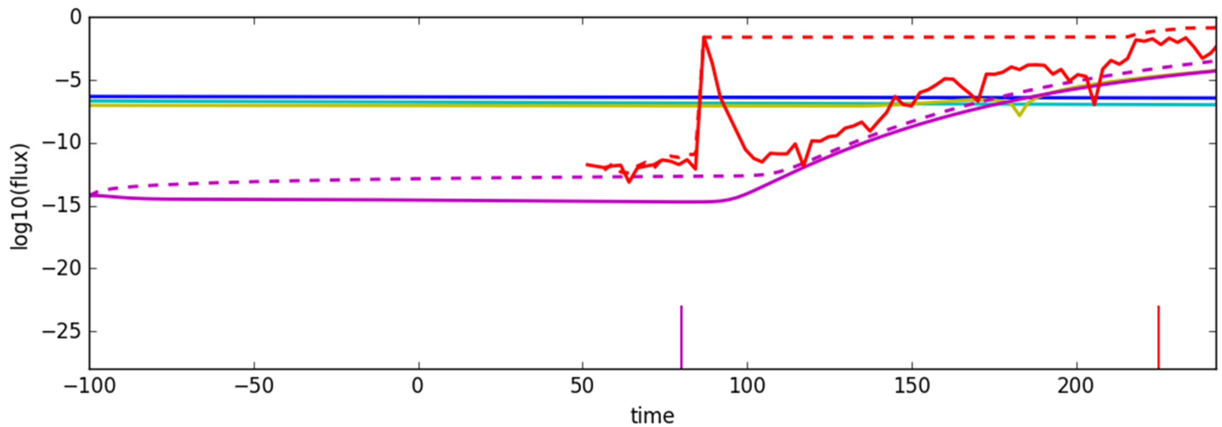
$v_e = 0.99$ and $v_{barrier} = 0$ $\delta x = 9$ $w = 5$ $V_{top} = 6.8$



$v_e = 0.8$ and $v_{barrier} = -0.9$ $\delta x = 9$ $w = 5$ $V_{top} = 6.8$



$v_e = 0$ and $v_{barrier} = -0.99$ $\delta x = 9$ $w = 5$ $V_{top} = 6.8$




Grazia