On the unique evolution of solutions to wave equations

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(joint work with Felicity Eperon & Harvey Reall)

Einstein's equations as evolution equations

(M, g) time-oriented Lorentzian manifold

Vacuum Einstein equations: $Ric_{\mu\nu}(g) = 0$.

Initial data: $(\Sigma, \overline{g}, k)$ where

- $(\Sigma, \overline{g})$ is 3-dimensional Riemannian manifold
- k is symmetric 2-covariant tensor field

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A *globally hyperbolic development* of an initial data set $(\Sigma, \overline{g}, k)$ is a solution (M, g) of the vacuum Einstein equations together with an embedding $\iota : \Sigma \hookrightarrow M$ such that

- $\iota(\Sigma)$ is a Cauchy hypersurface of M
- $\iota^*(g) = \overline{g}$
- $\iota^*(K) = k$, where K is the second fundamental form of $\iota(\Sigma)$ in M.

Theorem (Choquet-Bruhat, Geroch '69)

Given initial data for the vacuum Einstein equations there exists a unique (up to isometry) maximal globally hyperbolic development (MGHD).

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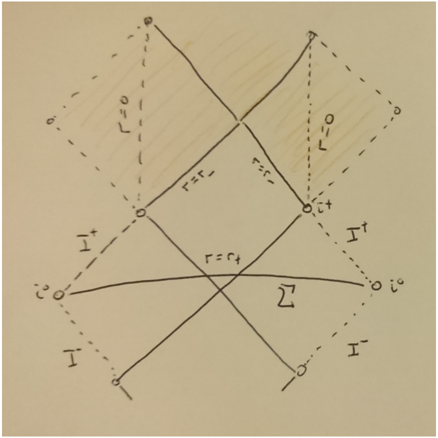

Summary

1. From evolutionary perspective of GR the MGHD is the fundamental object to study.
2. Necessary condition for existence of closed timelike curves inside black holes is that MGHD is generically extendible as a solution to Einstein's equation.
3. **As long as the development is globally hyperbolic, the evolution is unique.**

Wave equations on a fixed background

Setup

Quasilinear wave equation

$$g^{\mu\nu}(\phi, \partial\phi)\partial_\mu\partial_\nu\phi = F(\phi, \partial\phi)$$

- $\phi : \mathbb{R} \times \mathbb{R}^d \supseteq D \to \mathbb{R}$ smooth
- x^μ standard coordinates on $\mathbb{R} \times \mathbb{R}^d$
- $g^{\mu\nu}$ smooth Lorentzian metric
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Question: Does there exist a unique MGHD? Is, as long as the development stays globally hyperbolic, the evolution unique?

Answer: In general this is not the case!

Comparison to Einstein equations

If x^α wave coordinates (i.e. $\Box_g x^\alpha = 0$, where $\Box_g \psi = g^{\mu\nu}\nabla_\mu\nabla_\nu\psi$), then

$$0 = Ric_{\alpha\beta}(g) = -\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + N_{\alpha\beta}(g)(\partial g, \partial g)$$

becomes a quasilinear wave equation.

However, Einstein equations not defined on fixed background!

Counterexample

$\phi : \mathbb{R} \times \mathbb{R} \supseteq D \to \mathbb{R}$, coordinates: $(t, x) \in \mathbb{R} \times \mathbb{R}$

$$-(1 + (\partial_x \phi)^2) \cdot \partial_t^2 \phi + 2\partial_t \phi \partial_x \phi \cdot \partial_t \partial_x \phi + (1 - (\partial_t \phi)^2) \cdot \partial_x^2 \phi = 0$$

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- $1 + 1$-dimensional *scalar Born-Infeld equation*
- $F = 0$
- $$g^{\mu\nu}(\phi, \partial\phi) = \begin{pmatrix} -(1 + (\partial_x \phi)^2) & \partial_t \phi \partial_x \phi \\ \partial_t \phi \partial_x \phi & (1 - (\partial_t \phi)^2) \end{pmatrix}$$

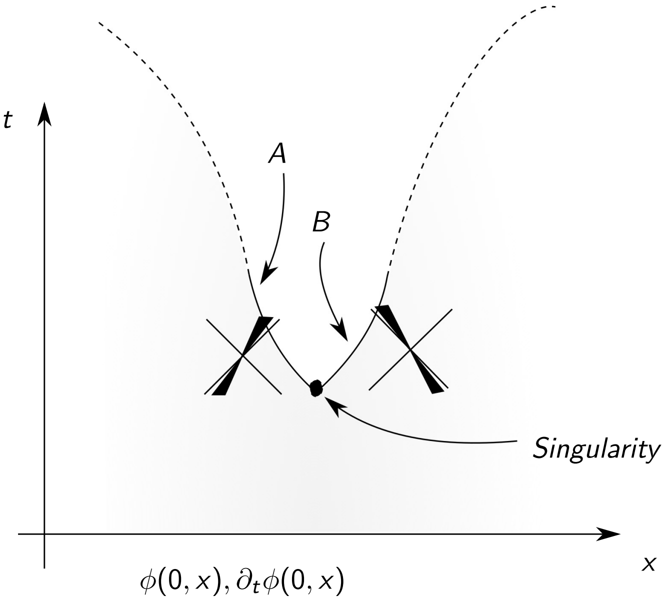
  if ϕ satisfies $1 + (\partial_x \phi)^2 - (\partial_t \phi)^2 > 0$, then g is a Lorentzian metric & g is subluminal.
- Gauge fixed equation for an infinite Nambu-Goto string with target space $2 + 1$-dim. Minkowski space
- Exactly solvable: Barbashov & Chernikov '60s

Theorem

For the equation

$$-(1+(\partial_x\phi)^2)\cdot\partial_t^2\phi+2\partial_t\phi\partial_x\phi\cdot\partial_t\partial_x\phi+(1-(\partial_t\phi)^2)\cdot\partial_x^2\phi=0$$

there exist two globally hyperbolic developments (GHDs) $\phi_a : D_a \to \mathbb{R}$, $\phi_b : D_b \to \mathbb{R}$ *of the same initial data posed on* $\{t=0\}$ *such that there exists* $x \in D_a \cap D_b$ *with* $\phi_a(x) \neq \phi_b(x)$.


t
A
B
Singularity
x
$\phi(0,x), \partial_t\phi(0,x)$

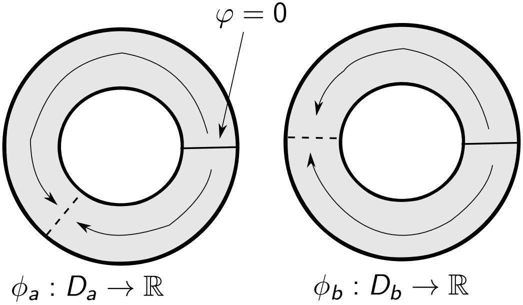
Important observations for later:

- $D_a \cap D_b$ is disconnected
- ∂D_a contains segment such that D_a 'lies on both sides of this segment' $\implies$ Solution blocks its own further evolution.

  (Same for D_b.)

This is qualitatively the same non-uniqueness mechanism as in spacetimes with closed timelike curves!

Toy-example: Let $\mathbb{S}^1 \times [-1,1]$ with metric $g = -d\varphi^2 + dx^2$ and consider the Cauchy problem for the wave equation $-\partial_\varphi^2 \phi + \partial_x^2 \phi = 0$ with initial data on $\varphi = 0$ and Dirichlet boundary conditions $\phi = 0$ on $x \in \{-1, 1\}$.



Uniqueness conditions for wave equations on a fixed background

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Theorem

Let $\phi_i : D_i \to \mathbb{R}$, $i = 1, 2$ be two GHDs of the same initial data. If $D_1 \cap D_2$ is connected, then $\phi_1 = \phi_2$ on $D_1 \cap D_2$.

Remark: For general quasilinear wave equations $D_1 \cap D_2$ is in general not connected – as shown by previous example of *subluminal* equation.

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Lemma

Assume there exists a vector field T such that T is timelike with respect to $g(\phi, \partial\phi)$ **for all possible** *$\phi, \partial\phi$. Let $\phi_i : D_i \to \mathbb{R}$, $i = 1, 2$ be two GHDs of the same initial data posed on a connected hypersurface S such that every maximal integral curve of T intersects S at most once. Then $D_1 \cap D_2$ is connected.*

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Combined, this gives uniqueness for example for *superluminal* quasilinear or semi-linear wave equations: choose $S = \{t = 0\}$ and $T = \partial_t$.

Can also show that a unique MGHD exists for such equations.

Definition

A GHD $\phi_1 : D_1 \to \mathbb{R}$ is called a maximal GHD (MGHD) iff there does not exist a GHD $\phi_2 : D_2 \to \mathbb{R}$ of the same initial data with $D_2 \supsetneq D_1$

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Theorem

Let $\phi_1 : D_1 \to \mathbb{R}$ be a MGHD such that for every $p \in \partial D_1$ there exists a neighbourhood V and a chart $\psi : V \to (-\varepsilon, \varepsilon)^{d+1}$ s.t. $\psi(\partial D_1 \cap V)$ is given by the graph of a continuous function $f : (-\varepsilon, \varepsilon)^d \to (-\varepsilon, \varepsilon)$ and all points below the graph lie in D_1 and all points above the graph lie in $\mathbb{R}^{d+1} \setminus D_1$. Then $\phi_1 : D_1 \to \mathbb{R}$ is the unique MGHD, i.e., any other GHD $\phi : D \to \mathbb{R}$ satisfies $D \subseteq D_1$.

Remarks:

- Shows that solution 'blocking' its own further evolution is the only non-uniquenss mechanism for smooth GHDs
- Result is teleological: First need to construct the whole MGHD – only then one can infer whether evolution was unique.
- Result applies to Christodoulou's study of the formation of shocks in 3-dimensional irrotational Euler equations.
  (D. Christodoulou, The formation of shocks in 3-dimensional fluids, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich (2007))