Finanziato dall'Unione europea NextGenerationEU
Ministero dell'Università e della Ricerca
Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA
ICSC Centro Nazionale di Ricerca in HPC, Big Data and Quantum Computing

Symmetric solutions for the N-body problem: a computational approach

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Spoke 3 General Meeting, Elba 5-9 / 05, 2024

Scientific Rationale

***N*-body problem:** predict the motion of *N* distinct particles under the action of the mutual gravitational interaction.

Poincaré conjecture: the **periodic solutions** are **dense** in the space of all possible bounded solutions.
The **chaoticity** of the *N*-body problem is one of the most famous problems in Celestial Mechanics, and the presence of (more and more complex) periodic solutions, which visit the many different regions of the phase space, goes in that direction.

Λ = class of all the possible periodic paths in the configuration space (functions from $\mathbb{R}/T\mathbb{Z}$ to a subset of $\mathbb{R}^{\dim \times N}$)
The solutions are found as **critical points** of the **Lagrangian action**

$$\mathcal{A} : \Lambda \to \mathbb{R}, \ \mathcal{A}(x) = \int_0^T K(x(t)) + U(x(t))\, dt$$

***G*-equivariance**: analytical tool that can be used to find **symmetric periodic solutions** (based on invariance with respect to a group action).

Technical Objectives, Methodologies and Solutions

Making use of the G-equivariance principle, search for highly complex periodic orbits:

➔ start by creating a **database** of “simple” symmetric solutions, using the **SymOrb** program. SymOrb is a program (originally written in FORTRAN, Python and GAP in the 2000s) returns symmetric solutions for the N-body problem obtained as (non necessarily minimal) critical point for the Lagrangian action. The orbit is obtained starting from random initial guesses and using different numerical methods (gradient descent, Newton,...) to find critical points;

➔ the database can be enriched by computing numerically, for each solution, **stability indicators** like the Floquet exponents (to estimate the rate of separation of close orbits) or the Morse index;

➔ the orbits in the database can be classified in terms of their shape and stability similarity (in Fourier space), to group together the ones with similar properties (to be done);

➔ starting from the above database, one can use optimization algorithms to produce **new orbits,** not necessarily periodic (for example, as combinations of the ones already found) (to be done).

Technical Objectives, Methodologies and Solutions

Example of database for 3 bodies in $\mathbb{R}^2$ and $\mathbb{R}^3$

In such case, there are 12 different symmetries. For every of them, one can start with different initial data and different methods to find critical points. The construction of the database is divided into two steps:

➔ **first set**: *minimal* critical trajectories (already done, over 1000 numerically computed solutions have been found);

➔ **second set**: critical trajectories which are *not necessarily minimal* (work in progress, using the results already obtained with step 1 to optimize the process; at present, over 2000 solutions have been found);

➔ **stability analysis**: once the solutions are found, their stability indicators can be numerically computed with suitable routines.

Note that we look for **collisional** orbits as well.

Timescale, Milestones and KPIs

- ➔ Translation of the SymOrb routine in Julia to make it **runnable** on modern laptops (September 2023 - May 2024)
- ➔ Use of SymOrb to construct a **big database** of periodic solutions of increasing complexity. Note that the detection of complex periodic solutions, from a theoretical point of view, could be particularly hard (step 1: September 2023 - April 2024; step 2: May 2024 - September 2024)
- ➔ Construction of routines to numerically compute **stability indicators** (Floquet exponents and Morse indices) optimization of the computational cost and comparison with the (few) known theoretical results (September 2023 - April 2024)
- ➔ **Classification** of the orbits of the database on the basis of their shape and stability similarity by clustering (by the end of 2024)
- ➔ Machine learning procedure (from September 2024):
  - ◆ use part of the data to train a parametric machine to find **new symmetric solutions**;
  - ◆ creation of new orbits, **not necessarily periodic**.
- ➔ Construction of **visualization tools** to display the solutions found both in 2D and 3D, with static and dynamic images (September 2023 - April 2024)

Accomplished Work, Results

➔ **SymOrb routine:** originally written in Fortran, Python and GAP, it has been refined and an optimised interface has been added.

➔ **Theoretical framework:** starting from the existing literature on the *G*-equivariant orbith of the *N*-body problem (*Ferrario - Terracini, 2004*), a detailed documentation on the theoretical framework has been written, and it has been the subject of a PhD course held in Spring 2023. After that, an extensive study on how to construct new, non trivial symmetry groups has been carried on.

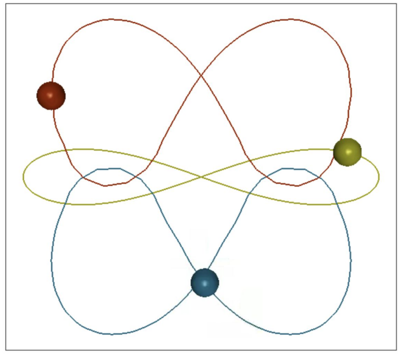
➔ **Database:** an extended database, has been produced for 3 bodies in 2 and 3 dimensions (see next slides).
The database of step 1 had to be rebuilt after correcting a bug in the code which increased a lot the computational time.

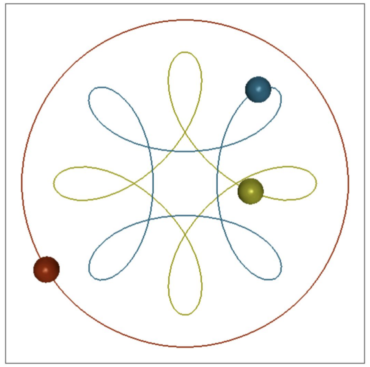
➔ **Stability indicators:** the routines for the numerical computations of stability indicators have been created (from scratch). They have been tested on several orbits and optimised. The results obtained numerically have been compared with the existing literature (see next slides).

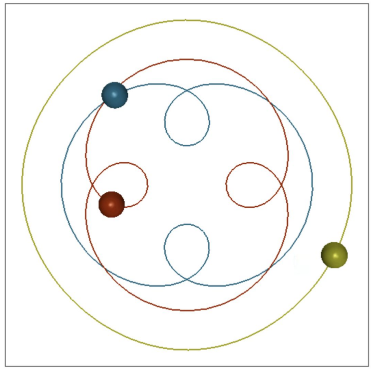
Accomplished Work, Results

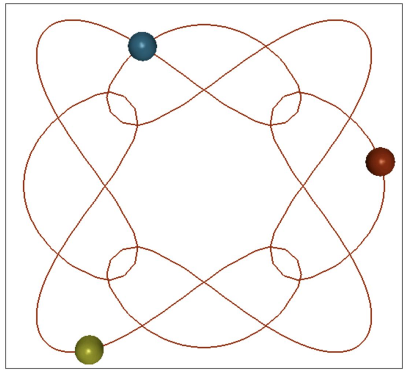
Examples of orbits from the database ~ 400 different trajectories

2D

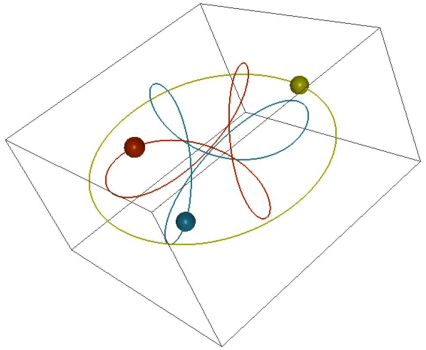


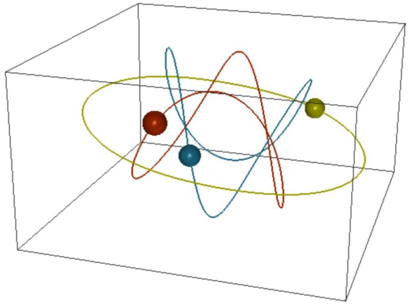






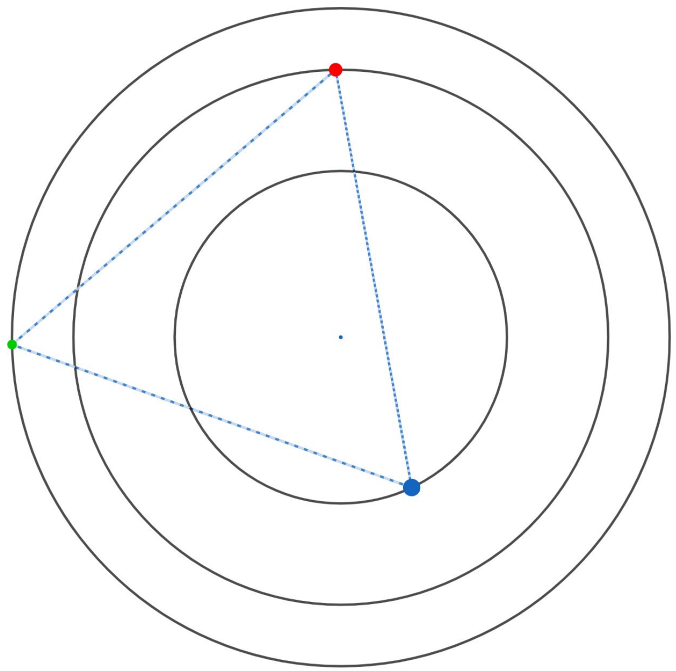
3D

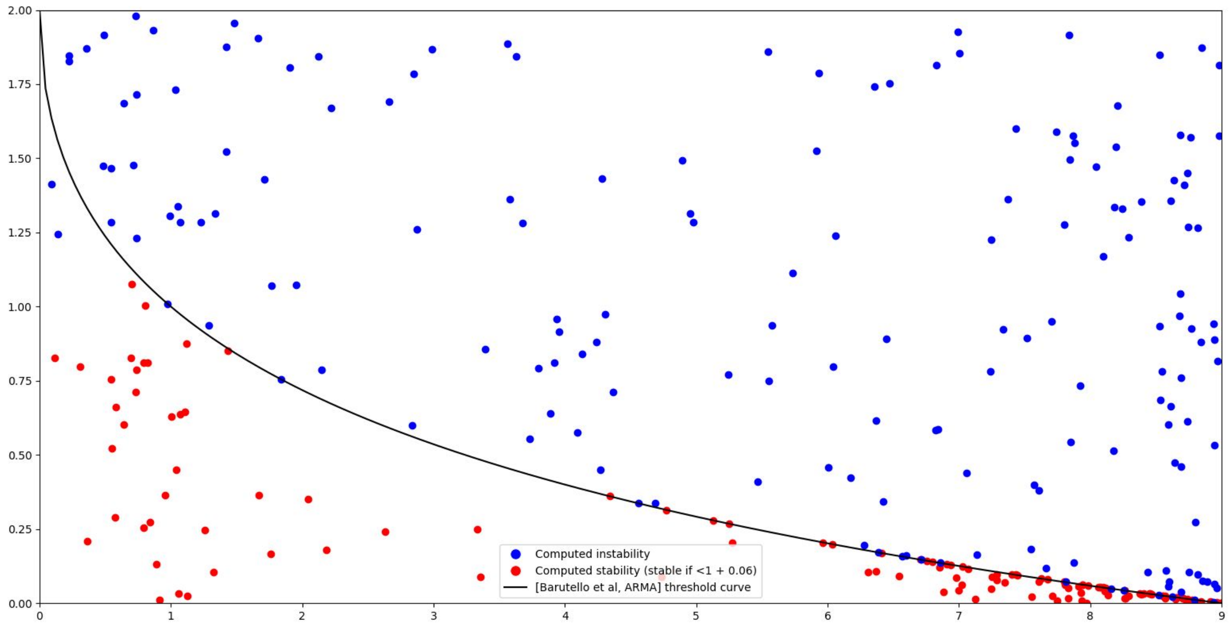




Accomplished Work, Results

Stability indicators: distinguish stable from unstable solutions.
Compute them analytically is particularly difficult, and it has been done only for few types of solutions. For random mass combinations, we compared the expected theoretical results to numerical ones.





NB: the application of this routine on a large database is already an interesting by-product of the project.

Next Steps and Expected Results

➔ **Database:** the database can be further enriched by considering a greater number of bodies in 2 or 3 dimensions, which could correspond to taking more and more complex symmetry groups. For such periodic trajectories, the stability properties can be derived.

➔ **Classification and visualization:** the wide database can be refined by grouping together the orbits with similar dynamic properties. A system to clearly visualize the similarities between orbits has to be developed.

➔ **New complex orbits:** search for strategies to explore the database and combine different periodic symmetric solutions, to the end of creating new solutions that "shadows" more than one simple trajectory, passing from one to another.

➔ **Parametrise the chaos:** find solutions that visits many different regions of the phase space. This is the ultimate objective of our analysis, since theoretically the chaotic behaviour is related to the presence of a solution dense in the phase space; such trajectory can be found as a limit of complex periodic solutions.

Thank you for your attention

For any questions/remarks, feel free to contact me at *irene.deblasi@unito.it*