Benchmarking Solar Simulations: An Analytical Solution for Non-Linear Diffusivity

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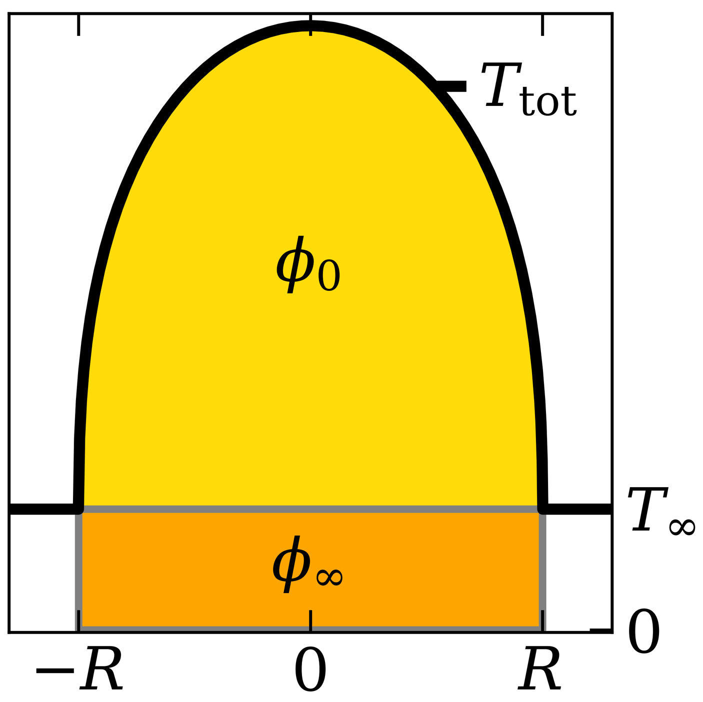
Abstract

Numerical simulations have proven invaluable in understanding the physics of the Sun. With increasing computing power available, we launch increasingly complicated multi-physics simulations. Every single physics module requires validation and we must understand the role of each of these physical processes. This work presents an analytical solution for non-linear diffusivity in 1D, 2D, and 3D. We will use it to benchmark the Spitzer conductivity module in the single and multi-fluid radiative MHD codes Bifrost and Ebysus. The solution is based on the self-similar solutions by Pattle, 1959, which required the diffusing quantity to be zero beyond a finite radius. We have surpassed this constraint, allowing for a small non-zero background value. This problem is highly relevant in the Solar atmosphere, where energy released in nanoflares or originating in the hot MK Corona diffuses to the much colder kK Photosphere. Beyond this use, the derivation and argumentation are general and can be applied to other non-linear diffusion problems.



Non-linear Diffusion

The goal here is to find an analytical solution for non-linear diffusion in s dimensions. Assuming radial symmetry, that can be written as

$$\frac{\partial T}{\partial t} = \frac{1}{r^{s-1}}\frac{\partial}{\partial r}\left(r^{s-1}D(T)\frac{\partial T}{\partial r}\right) \quad , \quad D(T) = KT^n$$

Ideal Solution, $T_\infty = 0$

Consider an initial instantaneous point source quantity ϕ_0 centered at $r = 0$, without any pre-existing background concentration, $T_\infty = 0$. The distribution will take the self-similar shapes derived by Pattle [1]. If we allow for a finite initial radius R_0, the shapes can be written as

$$T^{(0)}(r,t) = T_0\,(1+\chi t)^{-\frac{s}{sn+2}}\left(1-\frac{r^2}{R(t)^2}\right)^{\frac{1}{n}} \quad , \quad \text{if } r < R(t)$$

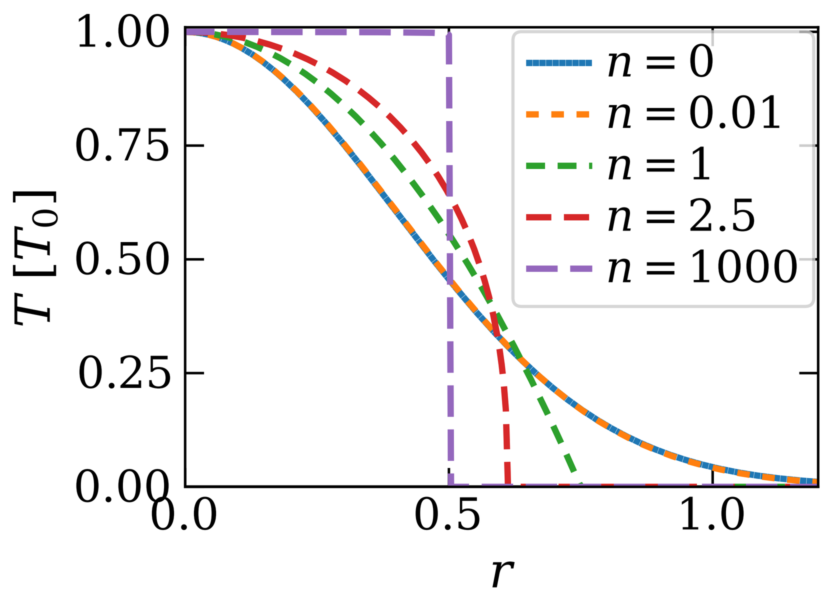
$$R^{(0)}(t) = R_0\,(1+\chi t)^{\frac{1}{sn+2}}$$

$$\chi = \frac{sn+2}{n}\frac{2D(T_0)}{R_0^2}$$

$$\phi_0\,G_{s,n} = T_0 R_0^s$$

In $s = 1$ dimensions, the self-similar shapes of different n and identical ϕ_0 look differently.

$n \to 0$: Gaussian ($D =$const)
$n \to \infty$: Step function



Perturbation Solution, $T_\infty > 0$

In many cases, the background concentration is nonzero, $T_\infty > 0$. Therefore, we search for a perturbation solution for a weak background concentration, $0 < T_\infty \ll T_0$. However, since T_∞ is dominant at $r > R(t)$, the perturbation cannot be taken in T.

In the interesting long-term time regime, $\chi t \gg 1$, it can be found that the ideal evolution depends only on ϕ_0, not T_0 or R_0. However, the ideal solution neglects the extra quantity ϕ_∞, illustrated in the abstract above. By considering this background quantity within $R^{(0)}(t)$ as a perturbation to the total quantity $\phi_{tot} = \phi_0 + \phi_\infty$, one gets

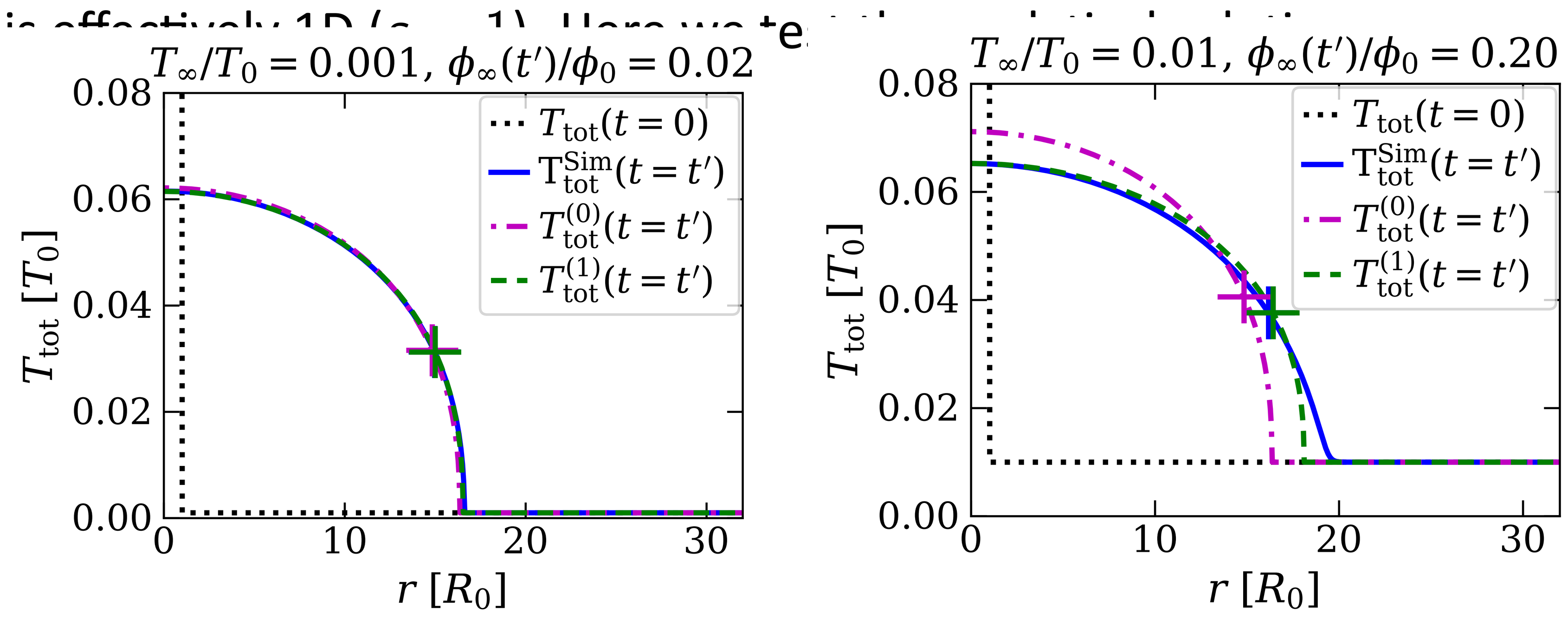
$$T^{(1)}_{tot}(r,t) = T_\infty + T_0\left[1+\chi t\left(1+\frac{\phi_\infty}{\phi_0}\right)^n\right]^{-\frac{s}{sn+2}}\left(1-\frac{r^2}{R(t)^2}\right)^{\frac{1}{n}}$$

$$R^{(1)}(t) = R_0\left[1+\chi t\left(1+\frac{\phi_\infty}{\phi_0}\right)^n\right]^{\frac{1}{sn+2}}$$

This perturbation solution assumes the distribution to maintain the ideal shape, on top of T_∞, but it widens faster due to the background.

Test of Spitzer Conductivity, $n = 5/2$

Spitzer conduction of electrons in a plasma is modeled by $n = 5/2$ [2]. In astrophysical plasmas, as in the Solar atmosphere, the conduction parallel to the magnetic field is typically dominant. Hence, the diffusion



The perturbation $T^{(1)}_{tot}$ is a better estimate of the simulation than the ideal solution $T^{(0)}_{tot} = T_\infty + T^{(0)}(r,t)$. The simulation does have wider tails than the analytical estimates for $\phi_\infty/\phi_0 = 0.20$.

How to Benchmark Your Simulation

When designing a test for your very own non-linear diffusion solver, for any exponent $n > 0$, you should keep in mind the following:

- $(\phi_\infty/\phi_0)^1 \ll 1$, if testing against the zeroth-order solution
  $(\phi_\infty/\phi_0)^2 \ll 1$, if testing against the first-order solution
- $\chi t \gg 1 \rightarrow R(t) \gtrsim R^{(0)}(t) = R_0(1+\chi t)^{\frac{1}{sn+2}} \gg R_0$
- $R_0 \gg \Delta r$, you must resolve the initial condition properly

Conclusion

We have successfully performed a first-order perturbation of the self-similar solutions by Pattle for non-linear diffusion with a nonzero background concentration, $T_\infty > 0$. In doing so, we have also verified that the non-linear solution is the limiting case of uniform diffusion.

The first-order perturbation is general, it works for all dimensions s and non-linear diffusion exponents n. We have also found constraints on the validity of both the ideal and first-order perturbation solutions.

The first-order perturbation agrees better with the truth. Still, it does not include the heavier tails, which occur when the perturbation quantity ϕ_∞ becomes comparable to the diffusing quantity ϕ_0.

References:
[1] R.E. Pattle, 1959, Quarterly Journal of Mechanics and Applied Mathematics **12**, p. 407
[2] L. Spitzer, 1962, Physics of Fully Ionized Gases, (Interscience, New York)