N-body chaos, discreteness effects and the continuum limit in self gravitating systems

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Outline

- N-body chaos in self consistent models and single particle dynamics
- Continuum limit
- Historical perspective
- Chaos in equilibrium and non-equilibrium $N-$body gravitational systems
- Single particle chaos
- Granularity and collisional systems, numerical simulations
- Summary and perspectives

Introduction: N-body chaos

$N-$body chaos as strong dependence of the initial conditions for systems described by Hamiltonians of the type

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}^2}{2m} + \sum_{j \neq i}^{N} V(||\mathbf{q}_i - \mathbf{q}_j||)$$

when the number of degrees of freedom (particles) N is "large". Quantified in terms of Lyapunov exponents.

$$\Lambda_{max} = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\mathbf{W}(t)}{\mathbf{W}_0} \right|,$$

Introduction: Continuum limit

The dynamics of $N-$body gravitational systems, due to the long-range nature of the $1/r^2$ force, is principally dominated by *mean field* effects rather than by inter-particle collisions for large N (e.g. as in galaxies where $N \approx 10^{12}$).

- Due to the extremely large number of particles it is often natural to idealize them in the continuum ($N \to \infty$, $m \to 0$) collisionless limit (particle behaves as a massless tracer) in terms of the Collisionless Boltzmann (Vlasov) Equation for the phase-space distribution $f(\mathbf{r}, \mathbf{v}, t)$

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \nabla \Phi \cdot \nabla_{\mathbf{v}} f = 0, \tag{1}$$

- Poisson equation $\Delta \Phi(\mathbf{r}) = 4\pi G \rho(\mathbf{r})$, where

$$\rho(r) = \int_{-\infty}^{+\infty} f(r, v) \mathrm{d}v \tag{2}$$

Introduction: time scales

- Free-fall dynamical time or crossing time

$$t_{\rm dyn} \propto \frac{r_s}{v_{\rm typ}} \approx 1/\sqrt{G\bar{\rho}}$$

- Collisional lifetime (Chandrasekhar 1943 two body relaxation time)

$$t_{2b} \propto \frac{v_{\rm typ}^3}{(Gm)^2 n \ln \Lambda} \approx \frac{N}{\log N} t_{\rm dyn} \quad \text{for} \quad \text{galaxies} \gg t_H \approx 13\text{Gyrs}$$

- CBE is valid until $t < t_{2b}$

Single (massive) particle dynamics

The motion of a particle in a potential Φ under the effect of dynamical friction and fluctuating gravitational force is

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla\Phi(\mathbf{r}) - \eta\frac{d\mathbf{r}}{dt} + F_W$$

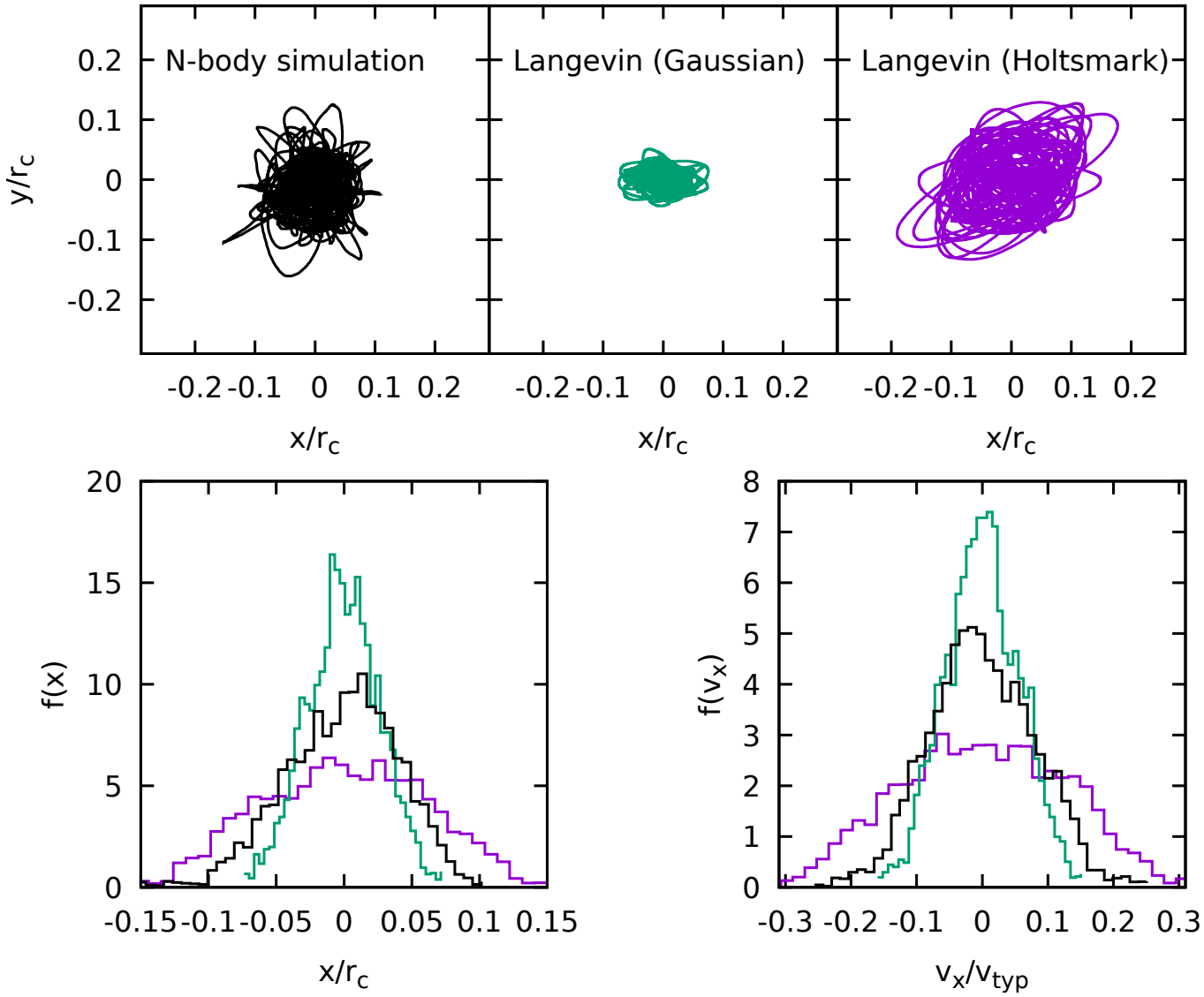
where:

$$\eta(r,v) = 4\pi G^2 m(M+m)\ln\Lambda\frac{\Psi(r,v)}{v^3}; \quad \Psi(r,v) = 4\pi\int_0^v f(r,v')v'^2\mathrm{d}v'$$

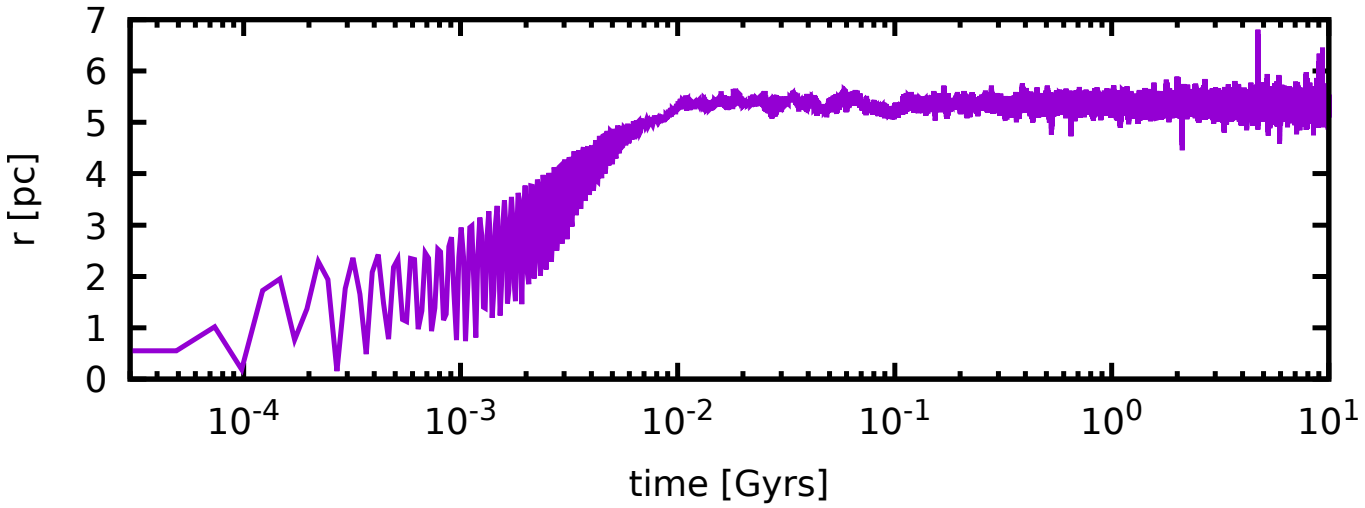
and the fluctuating force F_W is sampled from the Holtsmark (1911) distribution

$$H(F) = \frac{2}{\pi F}\int_0^\infty \exp\left[-\alpha(\xi/F)^{3/2}\right]\xi\sin(\xi)\mathrm{d}\xi; \quad \alpha = \frac{4n}{15}(2\pi Gm)^{3/2}$$

Single (massive) particle dynamics



Single (massive) particle dynamics



$M_{\rm gal} = 3 \times 10^{12} M_\odot$, $\gamma = 1.2$, $r_c = 3$ kpc, and a dark to visible matter ratio of ≈ 6, [parameters roughly corresponding to the case of M87. We observe that over a time of 10 Gyrs the SMBH reaches radii of the order of ≈ 6 pc, that is compatible with the off-centre displacement claimed for the SMBH of M87

The problem

- The CBE is and infinite dimensional (non-canonical) Hamiltonian system with infinite conserve quantities: the Casimir invariants

$$C(f,t) = \int_{\Omega} c(F)\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}$$

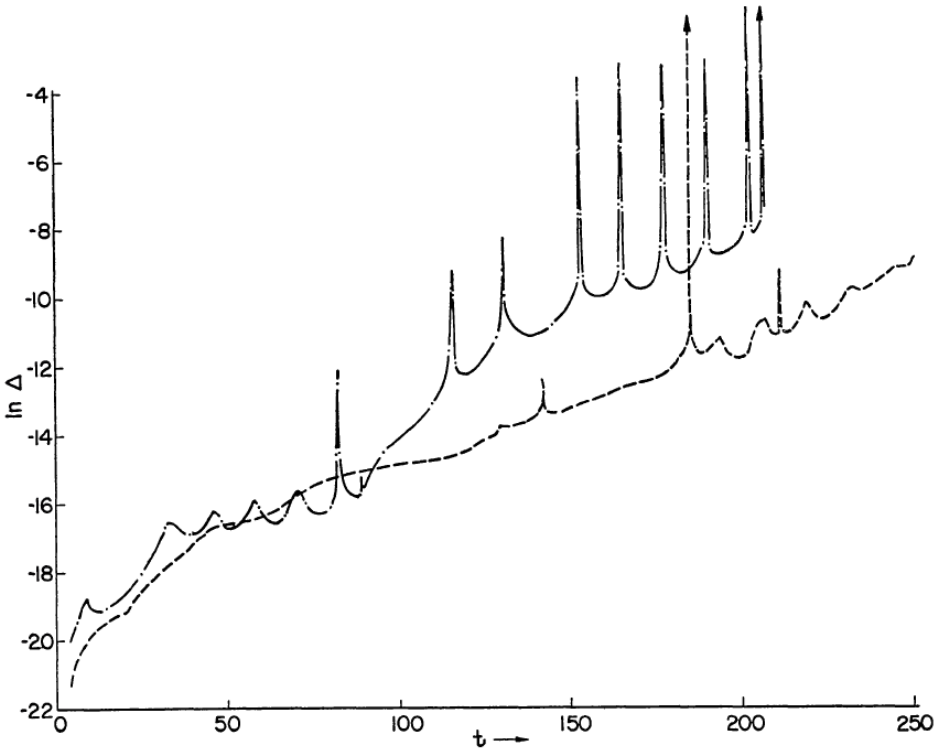
  where c is any continuous differentiable function.
- But we know from Celestial Mechanics for example that for $N \geq 3$ the $N-$body problem becomes non-integrable with more and more complex dynamics as N increases.

The problem

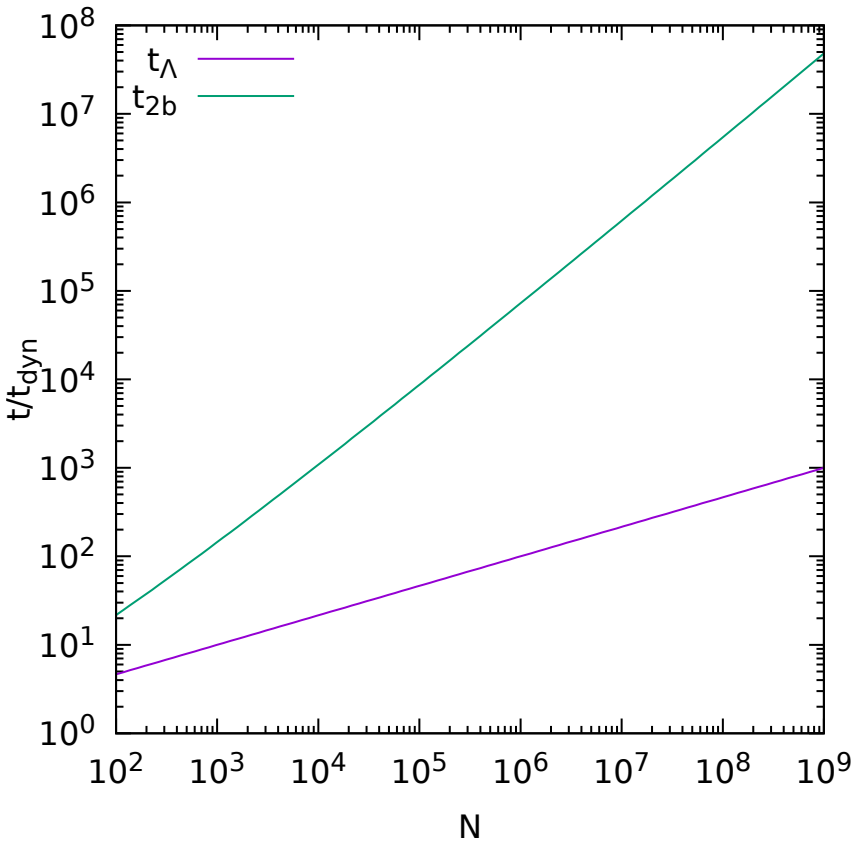
- Where is Chaos in the continuum (Vlasov) limit?
- What about models that have non-integral mean filed potentials in the continuum limit?
- Is the continuum limit meaningful after all?
- There are two points of view:
  1. Self consistent $N-$body dynamics, or orbit in R^6 (in the sense of analytical mechanics)
  2. Collective properties of families of single particle orbits (in the sense of Stellar/Celestial Dynamics)

What was done, so far

Miller (1964,1971) concluded that gravitational $N-$body simulations can not be idealized as a good representation of collisionless systems due to the exponential (Miller instability) growth of distances of nearby realizations



What was done, so far



"Collective relaxation time", (Gurzadyan & Savidy 1984,1986; Later work of Pettini and collaborators, Gurzadyan & Kocharyan 2009)

$$t_\Lambda \propto 1/\Lambda_{\max} \approx t_{\rm dyn} N^{1/3}; \quad \Lambda_{max} = \lim_{t\to\infty} \frac{1}{t} \ln\left|\frac{W(t)}{W_0}\right|,$$

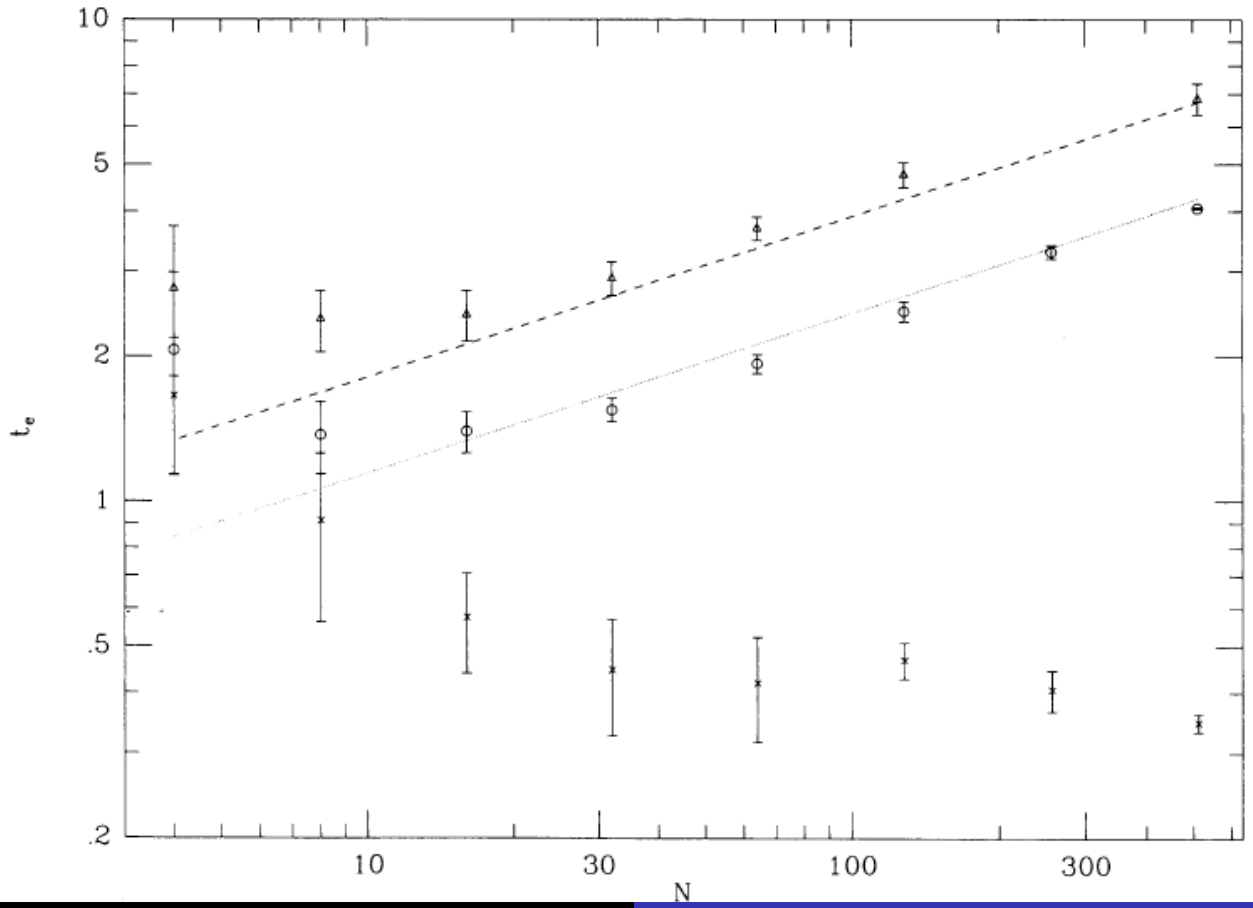
What was done, so far

- Typically $t_{\rm dyn} < t_\Lambda < t_{2b}$.
- t_Λ was obtained with differential geometry arguments and the statistics of gravitational field fluctuations (Holtsmark distribution, Chandrasekhar & von Neumann 1942,1943), as a quantity connected to discreteness effects rather than collisions.
- In a rather obscure paper Vesperini A& A 1992 suggested that t_Λ could be linked to a fast relaxation channel for GCs

What was done, so far

Goodman, Heggie & Hut (1993) and Hemsendorf & Merritt (2002) computed the instability growth rate

$$\mu = \frac{\ln \Delta(t_2) - \ln \Delta(t_1)}{t_1 - t_2} \tag{3}$$



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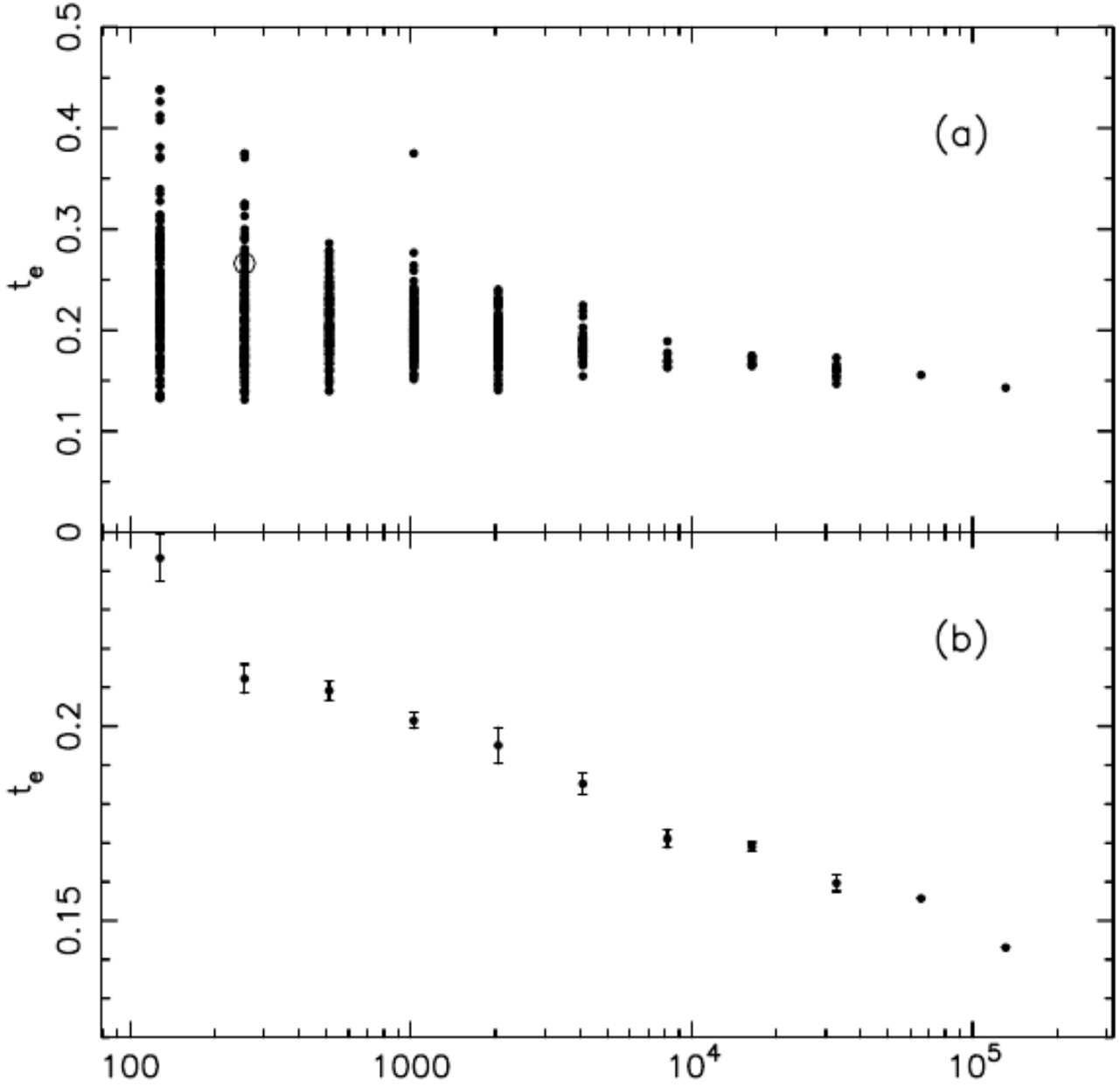
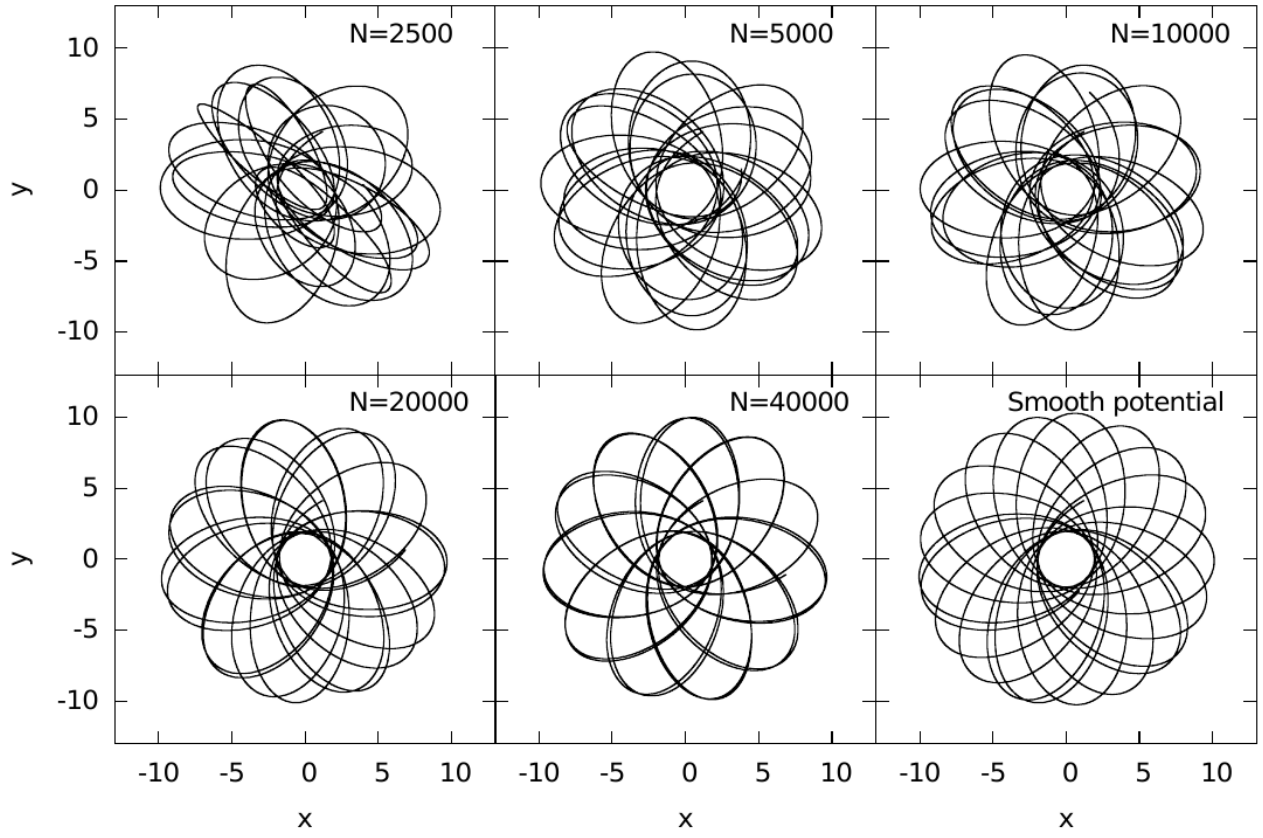
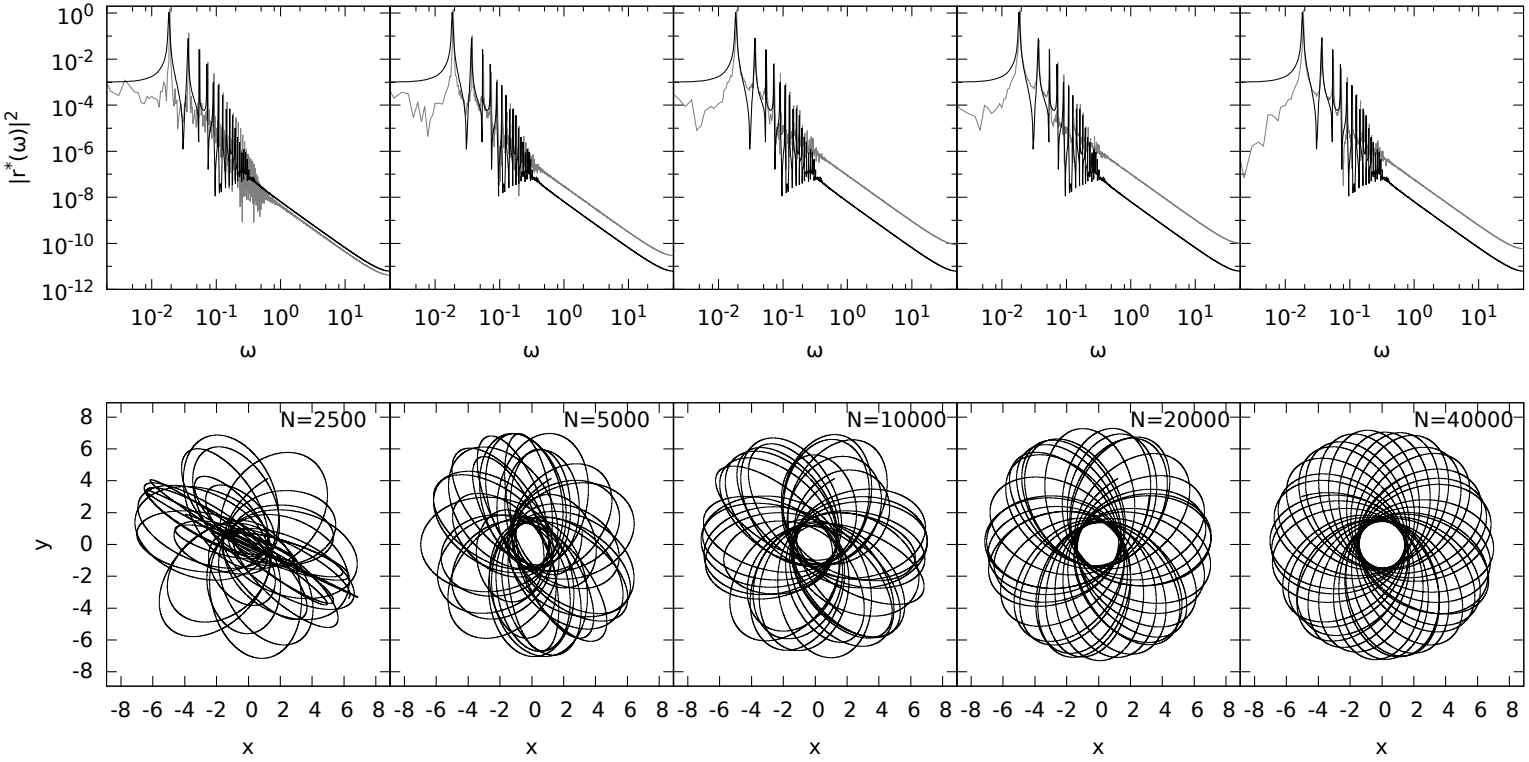


What was done, so far

Numerical studies from the mid 90s focusing on tracer particles suggested that $\lambda_{\max}$ is either constant with N or slightly increasing (Kandrup & collaborators 1995-2004). Individual particle orbits in frozen potential resemble more and more their continuuum limit counterparts as N increases



What was done, so far



What was done, so far

For models that admit chaotic (single particle) orbits in the continuum limit:

- Lyapunov exponents of tracer orbits increase with N
- The range of the chaos associated to the global potential decreases with N

Discreteness effects in equilibrium models

In Di Cintio & Casetti MNRAS 2019, IAU proceedings 2019 we integrate the equations of motion plus the variational equations with 4rd order symplectic integrator and compute the finite time Lyapunov exponents for different N at fixed density profile

$$\ddot{\mathbf{r}}_i = -Gm \sum_{j=1}^{N} \frac{\mathbf{r}_i - \mathbf{r}_j}{||\mathbf{r}_i - \mathbf{r}_j||^3}$$

$$\ddot{\mathbf{w}}_i = -Gm \sum_{j=1}^{N} \left[(\mathbf{w}_i - \mathbf{w}_j) - 3(\mathbf{r}_i - \mathbf{r}_j) \frac{(\mathbf{w}_i - \mathbf{w}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)}{||\mathbf{r}_i - \mathbf{r}_j||^2} \right] \frac{1}{||\mathbf{r}_i - \mathbf{r}_j||^3}$$

Self consistent anisotropic equilibrium models: Phase-space distribution

For a spherically symmetric, non-rotating continuum density profile $\rho(r)$ with anisotropy radius r_a and potential $\Phi(r)$, the anisotropic $f(Q)$ depends only on the quantity $Q = \mathcal{E} + J^2/2r_a^2$, where $\mathcal{E}$ is the energy, J is the angular momentum via the Osipkov (1979) - Merritt (1985) parametrization of the Eddington (1916) formula

$$f(Q) = \frac{1}{\sqrt{8}\pi^2} \frac{\mathrm{d}}{\mathrm{d}Q} \int_Q^{Q_{\mathrm{sup}}} \frac{\mathrm{d}\rho_a}{\mathrm{d}\Phi} \frac{\mathrm{d}\Phi}{\sqrt{\Phi - Q}} \tag{4}$$

$$\rho_a(r) = \rho(r) \left(1 + \frac{r^2}{r_a^2}\right) \tag{5}$$

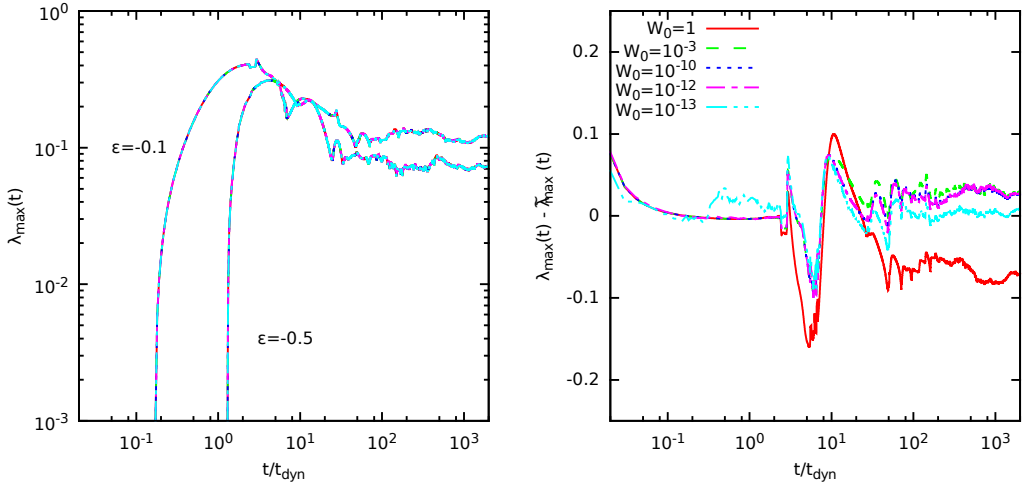
This can be used to built, starting from a smooth ρ, $N-$body realizations via standard rejection method.

N-body chaos in stable equilibrium models

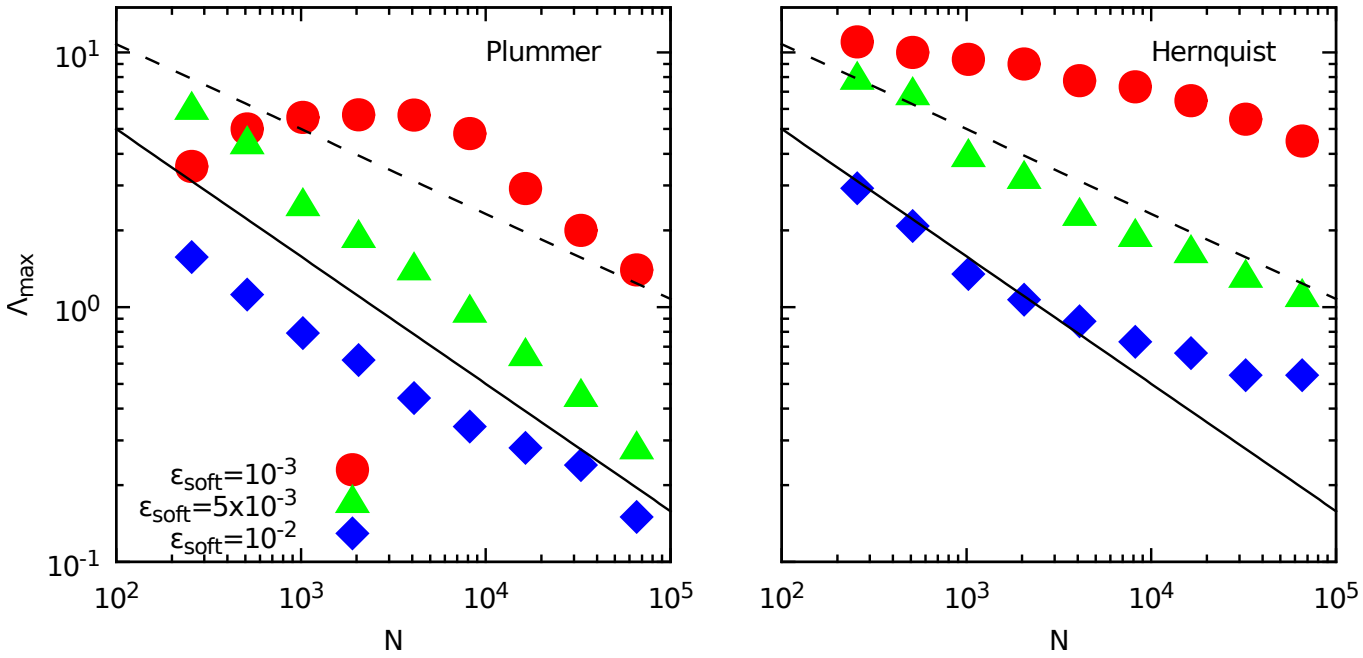
We compute the (numerical) maximal Lyapunov exponent with the standard Benettin, Galgani & Strelcyn Phys.Rev.A (1976) method as limit of

$$\lambda_{max}(t) = \frac{1}{k\Delta t}\sum_k \ln\left|\frac{W(k\Delta t)}{d}\right|,$$

for large t, where W is the norm of the 6N dimensional vector $\mathbf{W} = (\mathbf{w}_i, \dot{\mathbf{w}}_i, ...\mathbf{w}_N, \dot{\mathbf{w}}_N)$ for self consistent simulations and $\mathbf{W} = (\mathbf{w}, \dot{\mathbf{w}})$ for a tracer in a frozen $N-$body model. d is the norm at $t = 0$

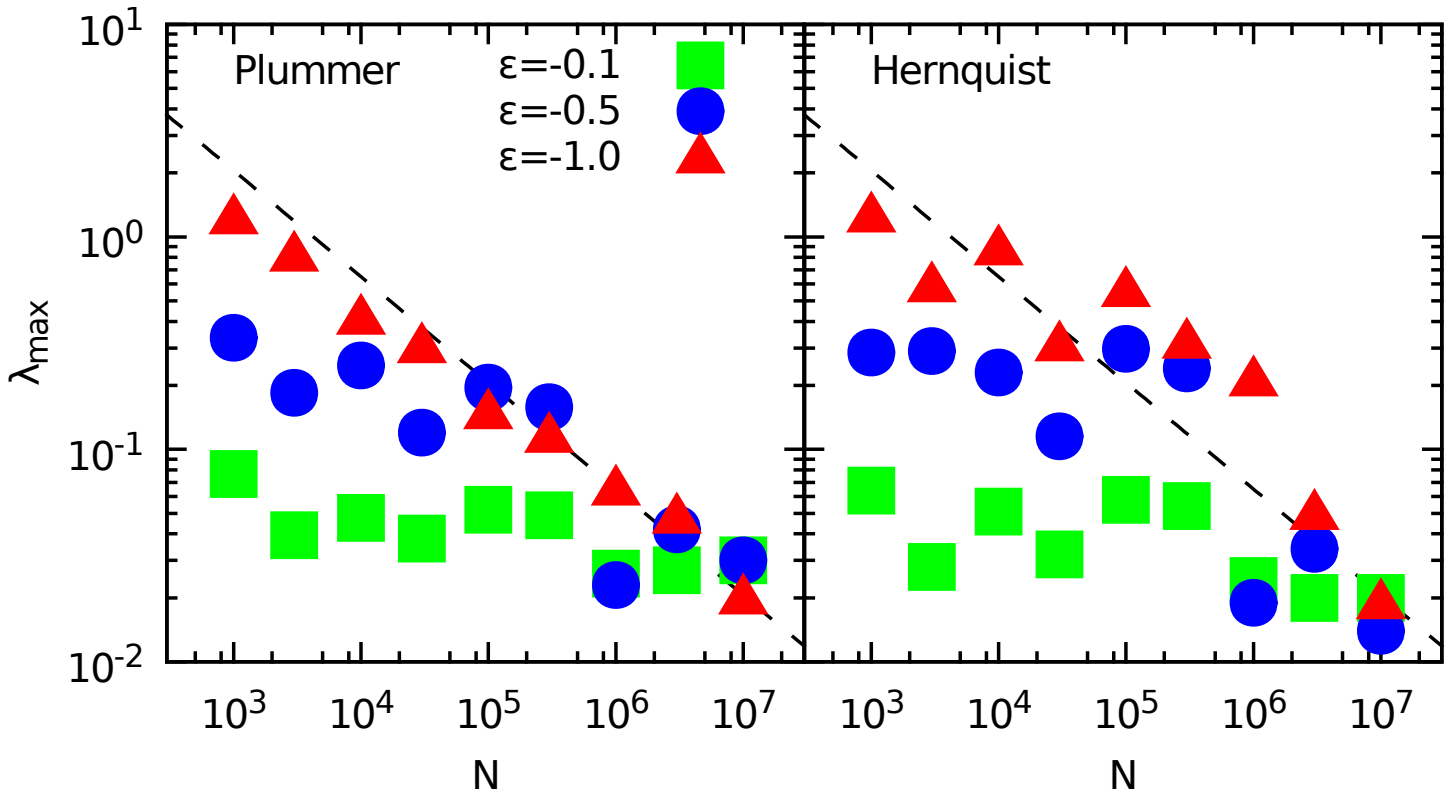


N-body chaos in stable equilibrium models



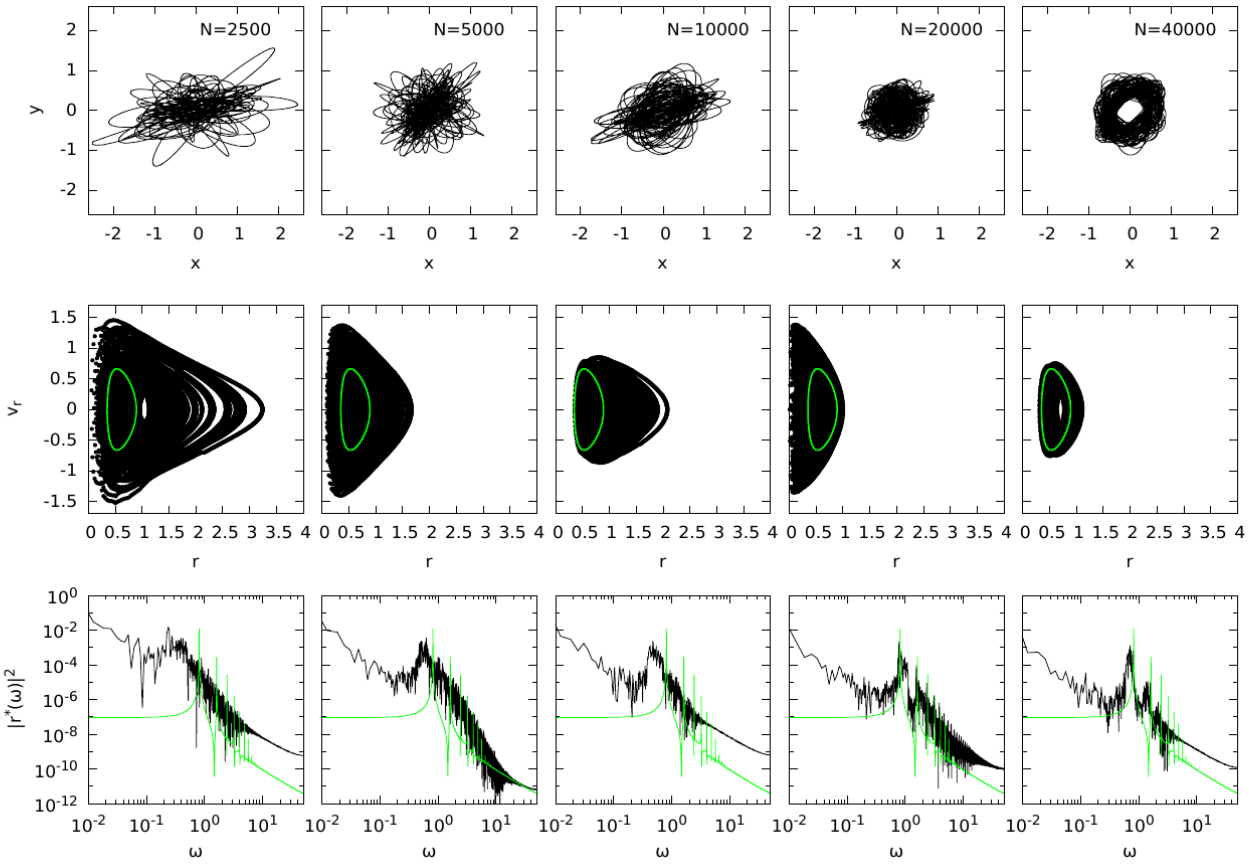
Central cusp leads to more chaos at large N. Flat core has a remarkable $N^{-1/2}$ trend. ϵ_{soft} has effect on the slope. $N^{1/3}$ scaling (Gurzdayan & Savvidy 1986) cannot be excluded.

N-body chaos in stable equilibrium models

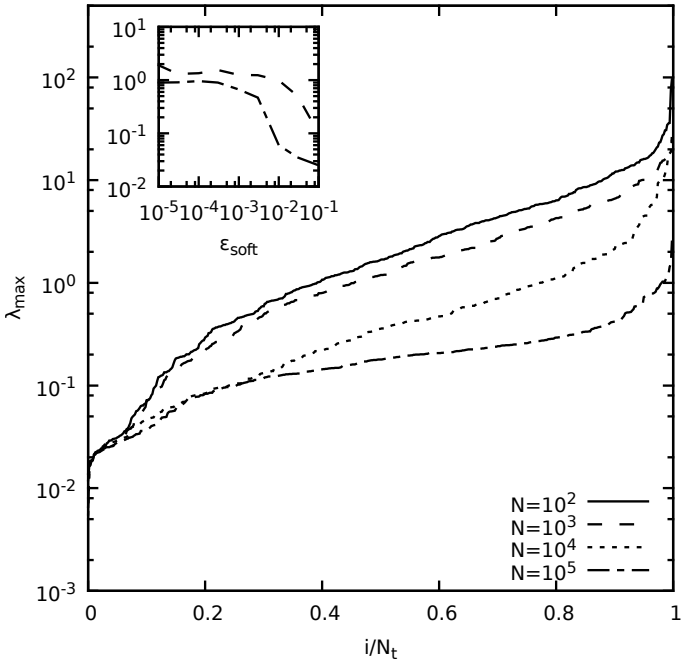


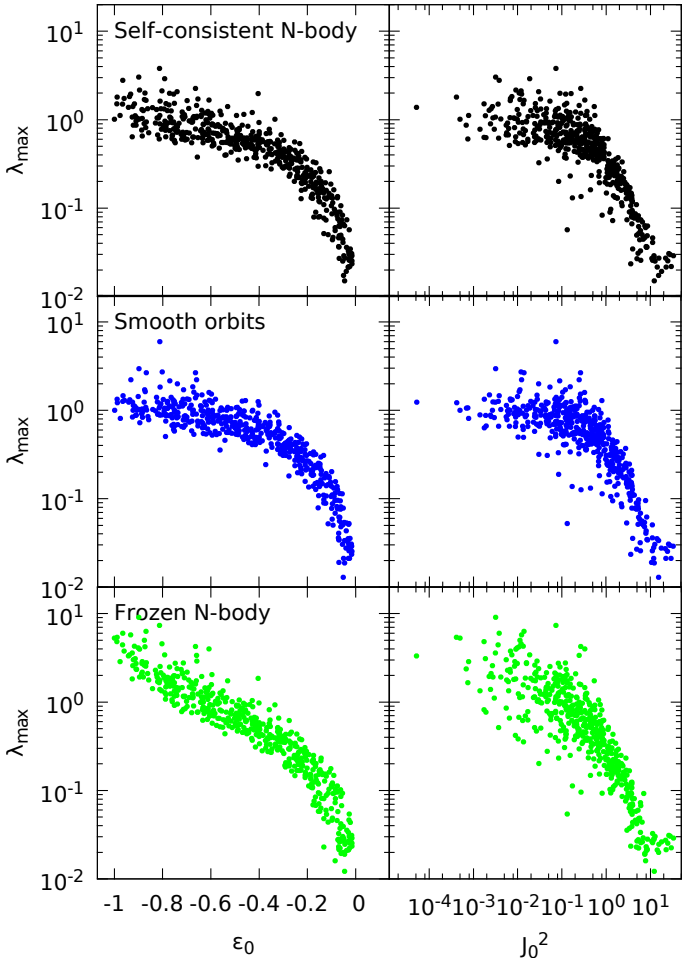
Strong energy dependence on N-scaling, $\lambda_{\max} \sim N^{-1/2}$ for strongly bound orbits

N-body chaos in stable equilibrium models



N-body chaos in stable equilibrium models





N-body chaos in unstable equilibrium models

1. Collisionless equilibrium systems with a significant fraction of the kinetic energy stored in low angular momentum orbits are violently unstable. The amount of radial orbits is quantified by introducing the Fridman-Polyachenko-Shukhman parameter

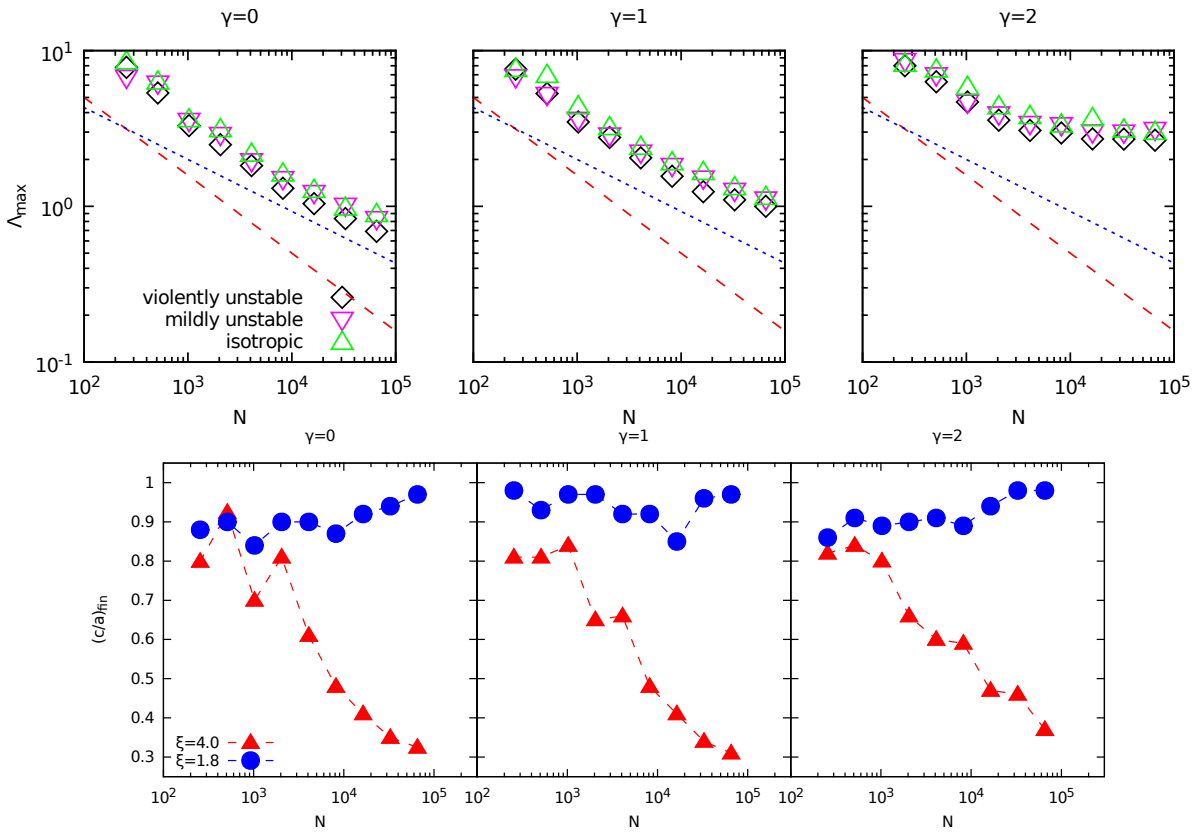
$$\xi = 2T_r/T_t, \tag{6}$$

   as function of the radial and tangential components of the (initial) kinetic energy T_r and T_t
2. For approximately $\xi > 1.5$ Newtonian systems appear to be unstable, leading to triaxial end-states.

N-body chaos in unstable equilibrium models

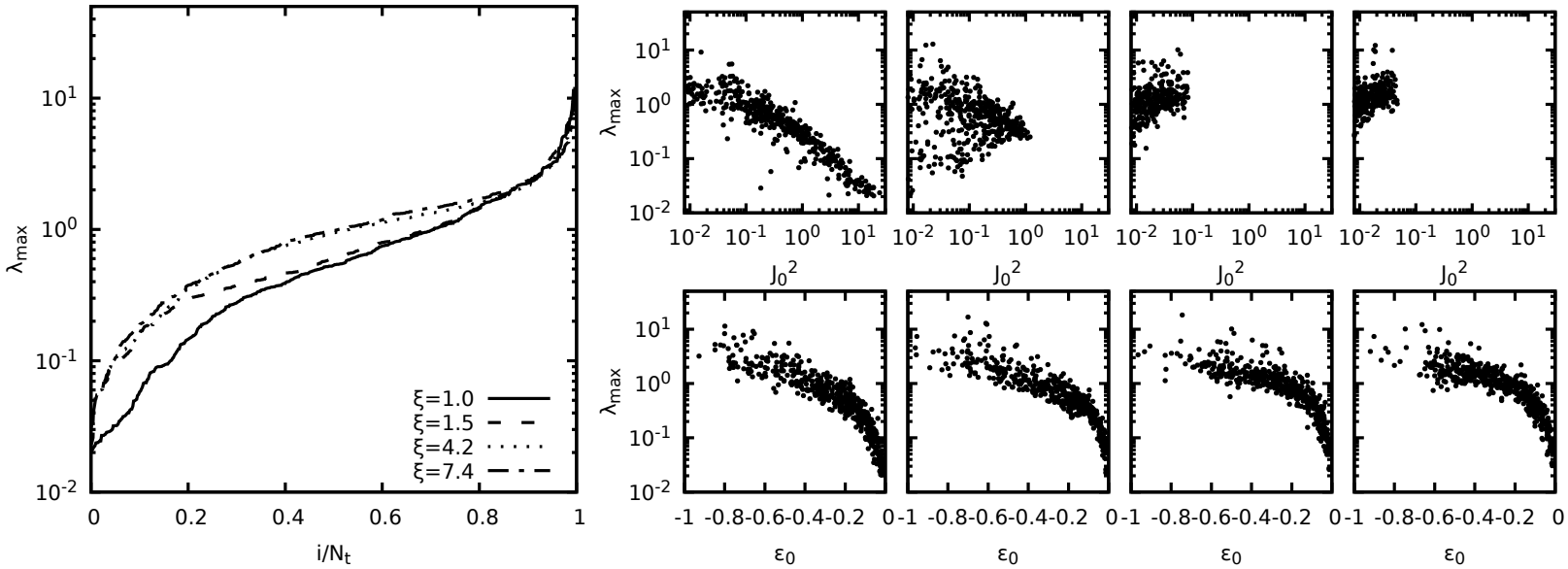
1. Analytical stability results exist for the isotropic case. It is also known that phase-space distribution functions with $\mathrm{d}f(\mathcal{E})/\mathrm{d}\mathcal{E} \leq 0$ correspond to stable systems (Antonov 1968 theorem)
2. ROI is triggered by particles with orbital frequencies close to satisfying the condition $\Omega_P \equiv 2\Omega_\nu - \Omega_r \simeq 0$, where Ω_ν is the azimuthal frequency, Ω_r the radial frequency and Ω_P the precession frequency (Palmer 1985, Palmer & Papaloizou 1987)
3. Once a small non-spherical density perturbation is formed in a system dominated by low Ω_P orbits, it will grow more and more, as more and more particles tend to accumulate to it.

N-body chaos in unstable equilibrium models

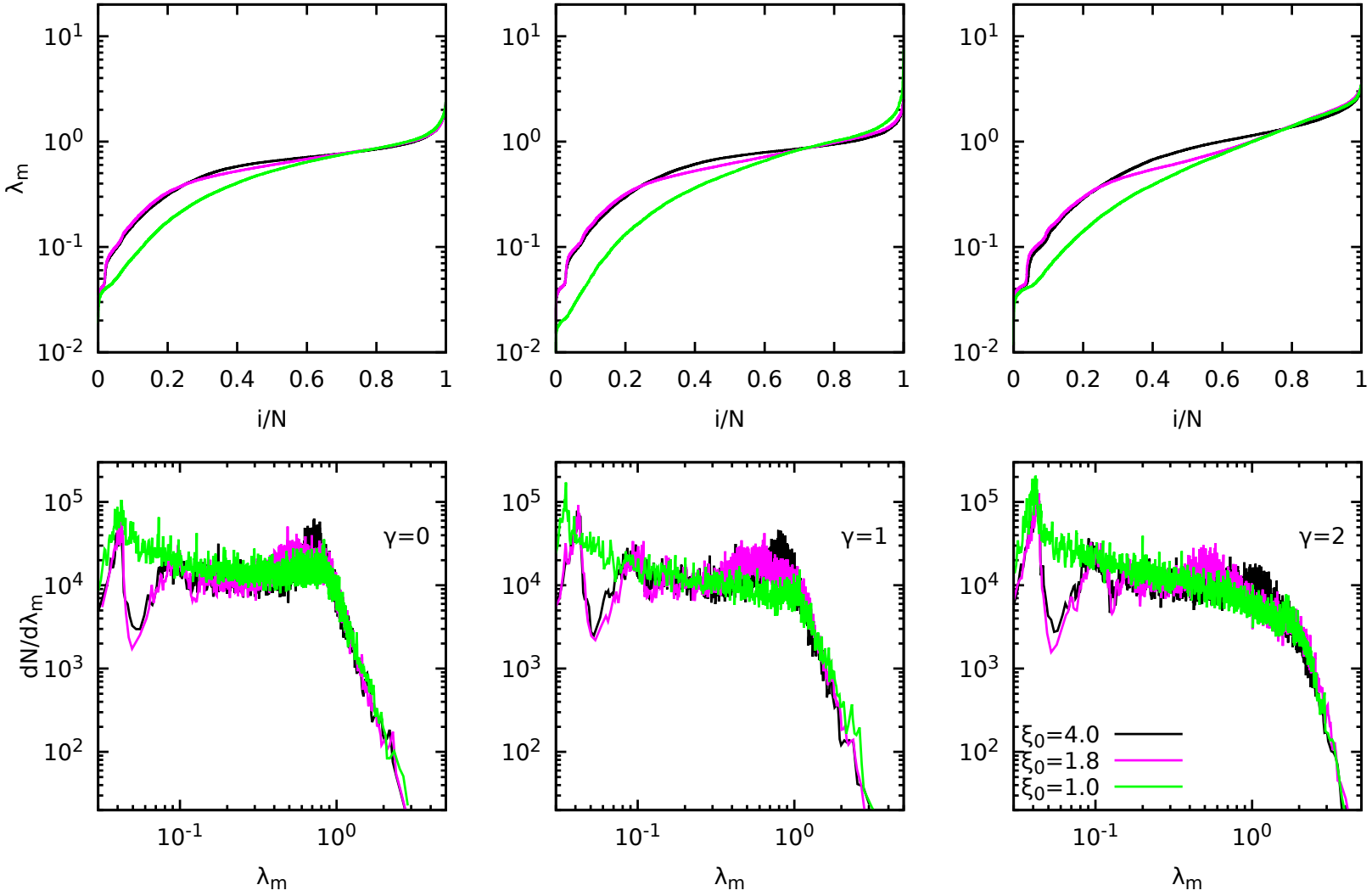


Anisotropy of parent f has little influence on $\Lambda_{\max}$, though larger systems are more triaxial in the end

N-body chaos in unstable equilibrium models

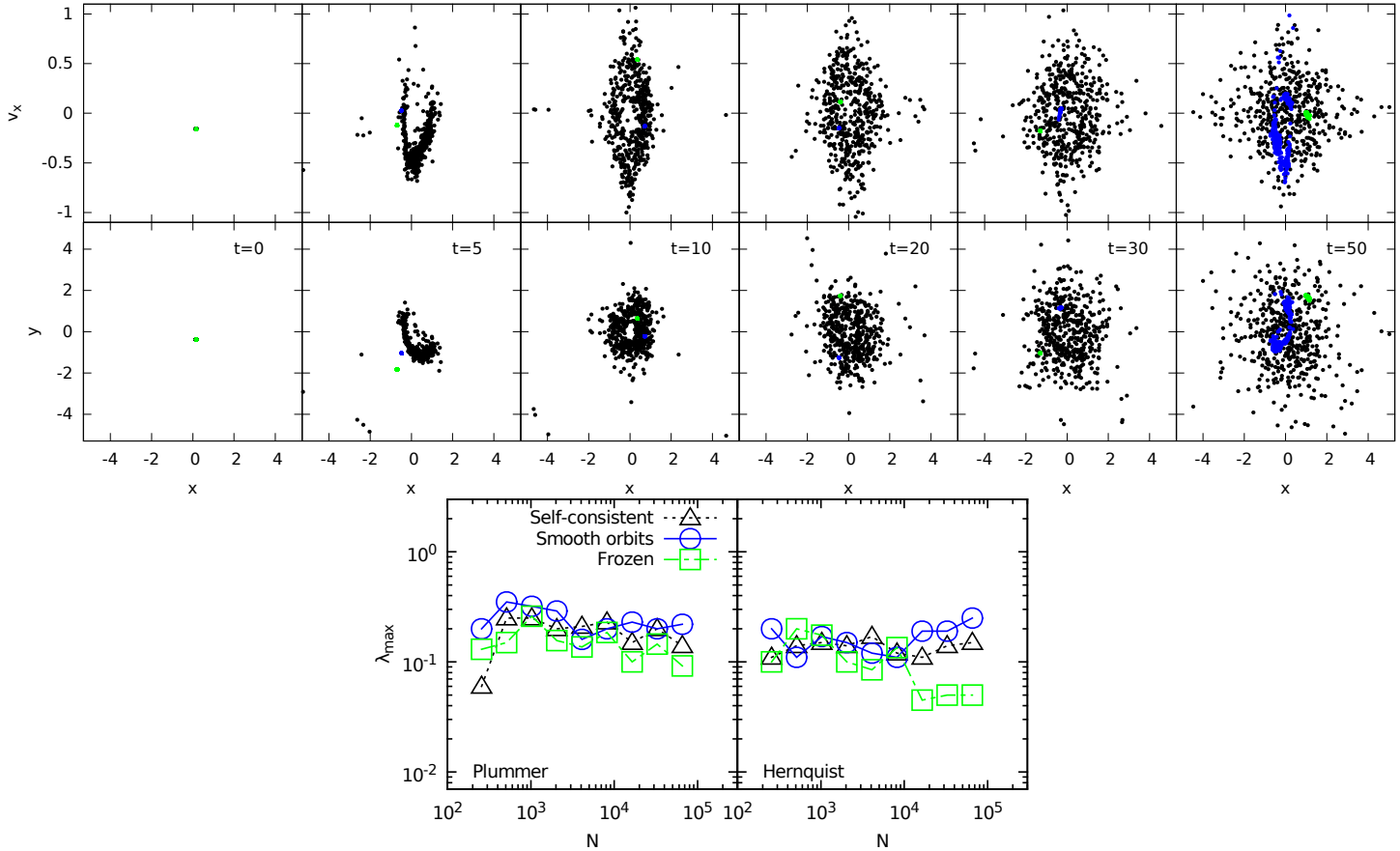


N-body chaos in unstable equilibrium models



Phase space diffusion

Tracers in self-consistent models diffuse. λ is weakly dependent on whether the model is "live or not"

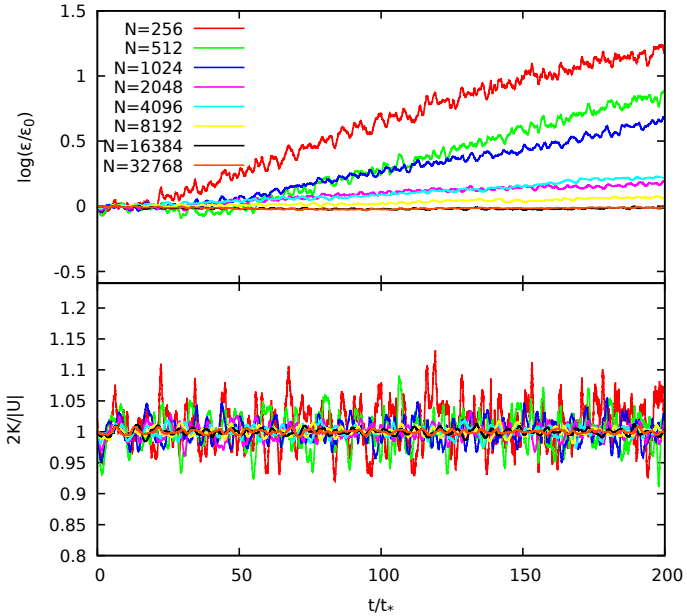


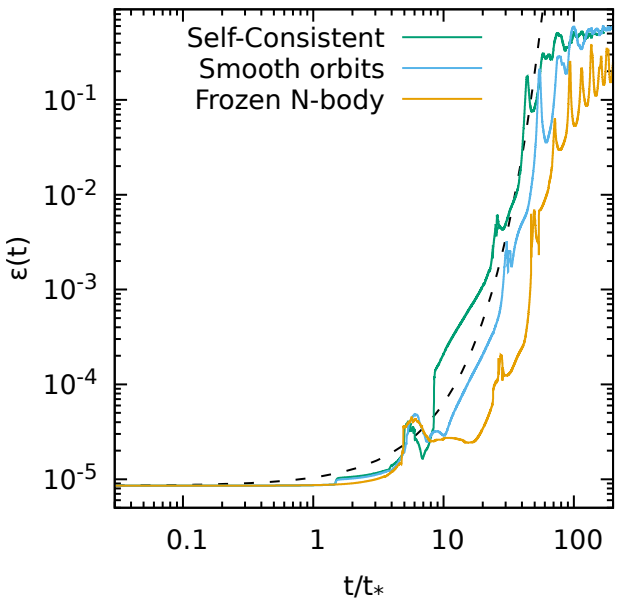
Phase space diffusion

The diffusion in phase-space is characterized by the emittance

$$\epsilon = (\epsilon_x \epsilon_y \epsilon_z)^{1/3}, \quad \epsilon_i = \sqrt{\langle r_i^2 \rangle \langle v_i^2 \rangle - \langle r_i v_i \rangle^2},$$

where $\langle ... \rangle$ indicate ensemble averages.





Wrap-up

- We verified the $N^{1/3}$ scaling of the Gurzdayan-Savidy relaxation scale
- Orbits in frozen models and active self consistent models have (obviously) different mixing properties and have, in general, different maximal Lyapunov exponents. $\Lambda_{\max}$ depends more on N in frozen systems.
- ROI is not associated to N-body chaos but rather to individual orbits Lyapunov times
- The continuum limit might be valid below t_Λ rather than below t_{2b}

Collisional vs collisionless $N-$body systems

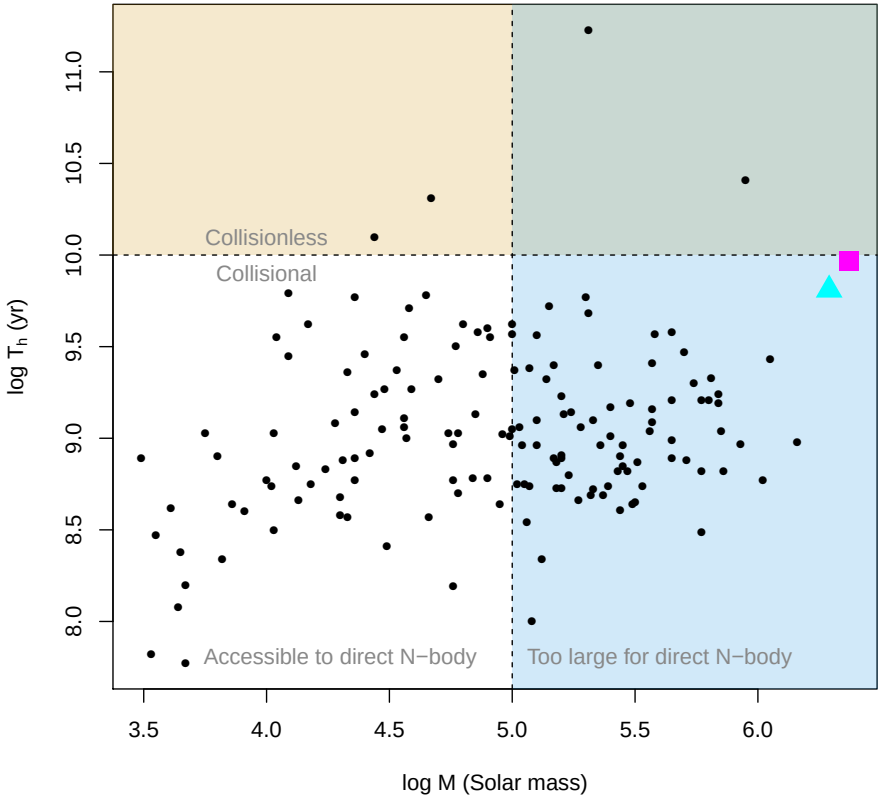
In stellar dynamics one typically defines two time scales the dynamical (or crossing) time $t_{\rm dyn}$ and the two body relaxation time t_{2b}:

$$t_{\rm dyn} = \sqrt{r_s^3/GM}; \quad t_{2b} \propto \frac{v_{\rm typ}^3}{G\langle m\rangle \rho \ln \Lambda} \approx t_{\rm dyn}\frac{N}{\ln N} \tag{7}$$

When $t_{2b} > t_H = 13$ Gyrs the system is said to be collisionless (i.e. the granular nature of the stellar distribution is effectively irrelevant for the dynamics of individual stars) otherwise collisional

Collisional vs collisionless $N-$body systems

Only a sub-set of GCs are accessible with state-of-the-art honest direct $N-$body



How to model collisional & weakly collisional systems

- Direct force calculation, but time scales with $O(N^2)$
- Fokker-Planck methods (Henon 1969)
- Hybrid PIC-Montecarlo methods: Cartwright, Verboncoeur & Birdsall, Phys.Plasm. **7**, 3252 (2000); Vasiliev, MNRAS **446**, 3150 (2015)
- Hybrid Particle-Mesh Direct force cell by cell. (see Hockney & Eastwood 1988)
- Multi-particle collision scheme plus standard PIC or particle-mesh (Di Cintio and collaborators 2015-2021)

The multi-particle collision method

- It actually comes from fluid dynamics: Malevanets & Kapral J.Chem.Phys. **112**, 7260 (2000)
- Collision are *stochastic* but preserve total momentum, kinetic energy and number of particles, i.e.:

$$P_i = \sum_{j=1}^{N_i} m_j v_{j,\mathrm{old}} = \sum_{j=1}^{N_i} m_j v_{j,\mathrm{new}} = \sum_{j=1}^{N_i} m_j (a_i w_j + b_i);$$

$$K_i = \sum_{i=1}^{N_j} m_j \frac{v_{j,\mathrm{old}}^2}{2} = \sum_{j=1}^{N_i} m_j \frac{v_{j,\mathrm{new}}^2}{2} = \sum_{j=1}^{N_i} m_j \frac{(a_i w_j + b_i)^2}{2},$$

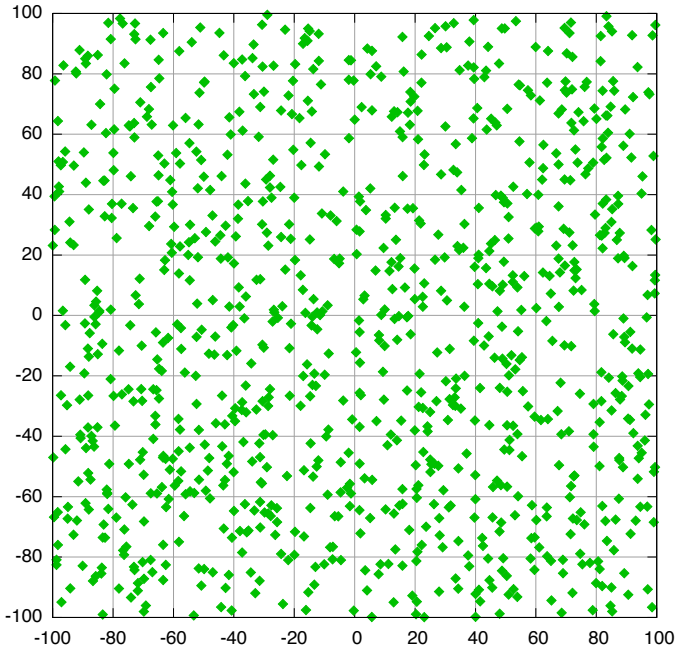
- It is a grid based method scaling as $N \log N$

The multi-particle collision method

The system is coarse-grained on a grid, then in each collision cell a stochastic rotation of the velocity vectors takes place:

$$\mathbf{v}_i(t+\Delta t) = \mathbf{u}_i(t)+\delta\mathbf{v}_{i,\perp}(t)\cos(\alpha)+(\delta\mathbf{v}_{i,\perp}(t)\times\hat{\mathbf{R}})\sin(\alpha)+\delta\mathbf{v}_{i,\|}(t),$$

where $\mathbf{R}$ is a random axis and $\mathbf{u}$ the c.o.m. speed and



$$\sin(\alpha) = -2AB/(A^2+B^2)$$
$$\cos(\alpha) = (A^2-B^2)/(A^2+B^2)$$
$$A = \sum_{i=1}^{N_c} [\mathbf{r}_\times(\mathbf{v}_i-\mathbf{u})]\,|_z$$
$$B = \sum_{i=1}^{N_c} \langle \mathbf{r}_i; (\mathbf{v}_i-\mathbf{u})\rangle$$

standard propagation with second order Verlet scheme.

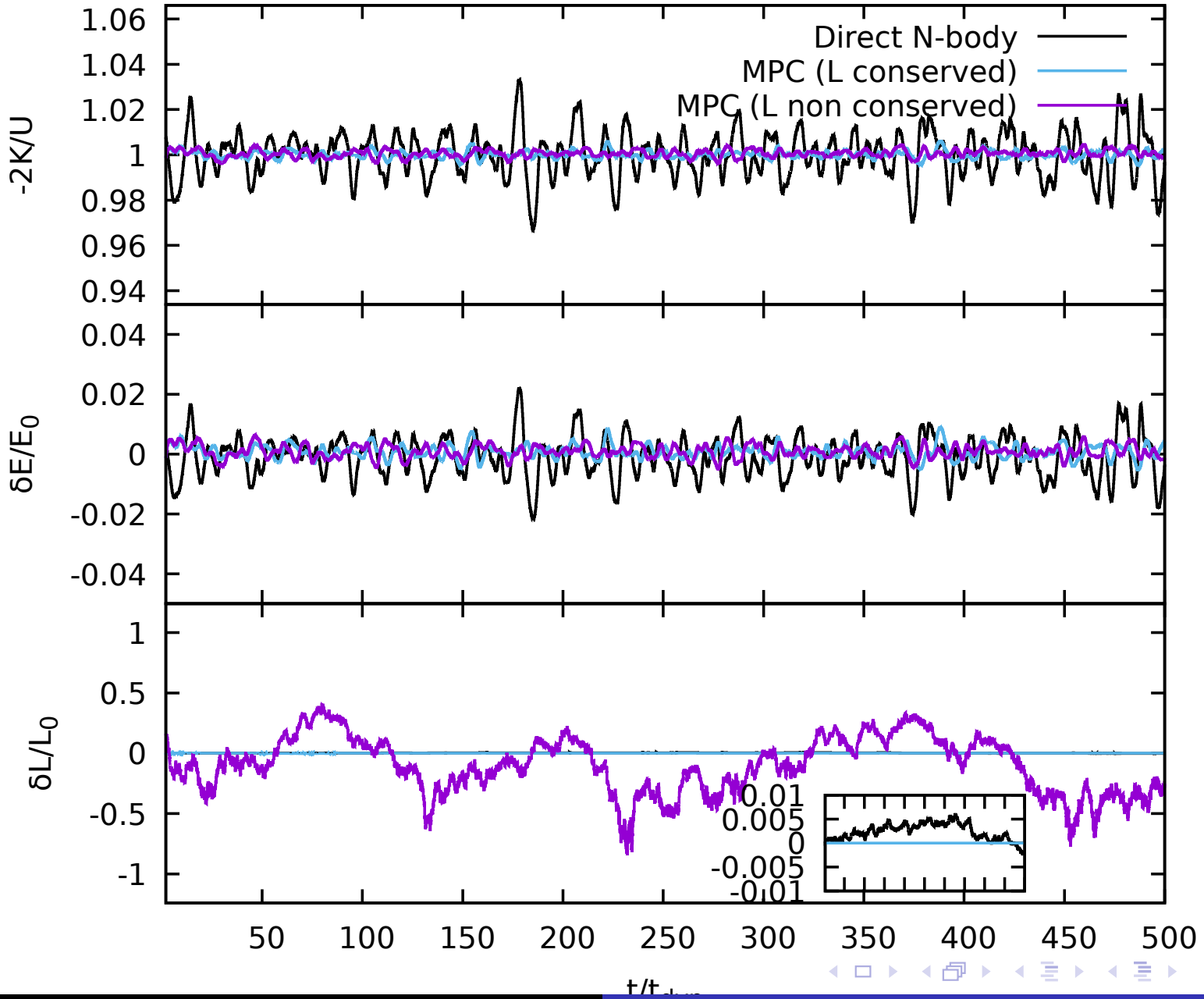
MPCDSS

- Long-range part of Gravitational interaction treated with Particle-mesh or PIC (mean field)
- Collisions implemented on the sub-mesh scale with MPC with a cell-dependent collision probability (Bufferand, Ciraolo et al.(2017), Di Cintio et al 2017,2021)

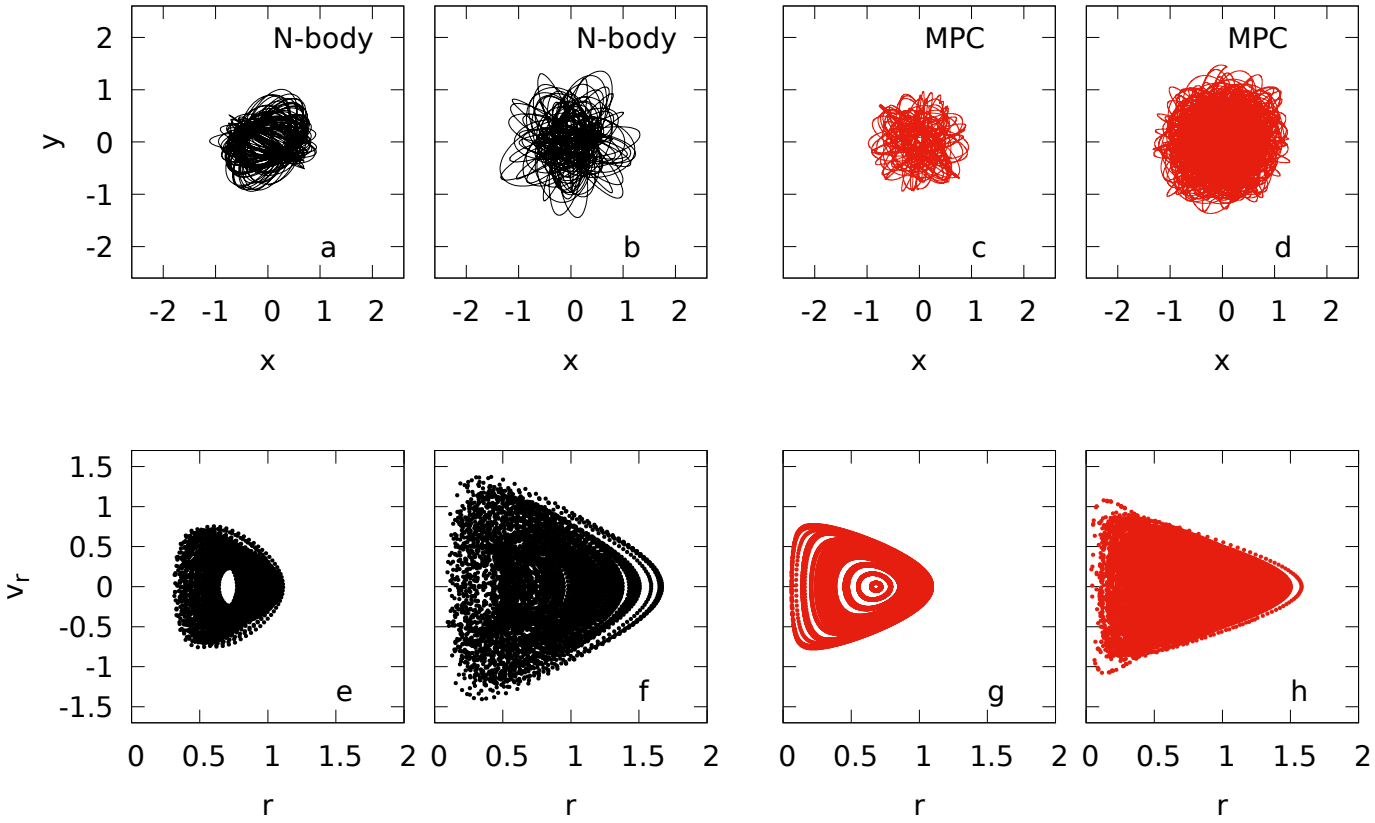
$$p_i = \mathrm{Erf}\left(\beta \frac{\Delta t 8\pi G^2 \bar{m}_i^2 \bar{n} \log \Lambda_i}{\sigma_i^3}\right)$$

  compared to a random number p_* extracted with uniform probability in $0 \leq p_* \leq 1$.
- In inhomogeneous systems collision happen only where and when p_i is large, provided that the cell size is smaller than the mean free path.

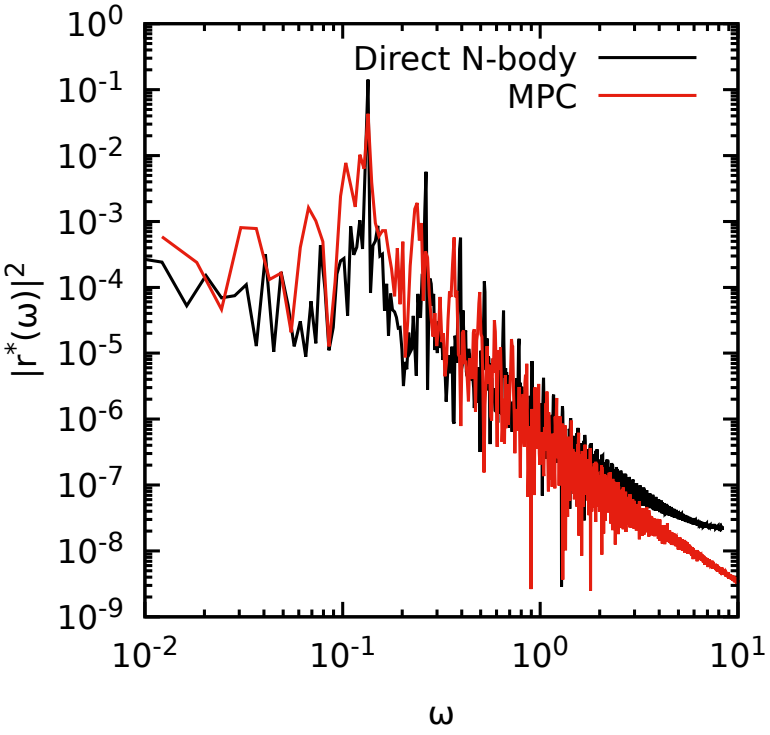
Comparison with direct N-BODY: Conservation

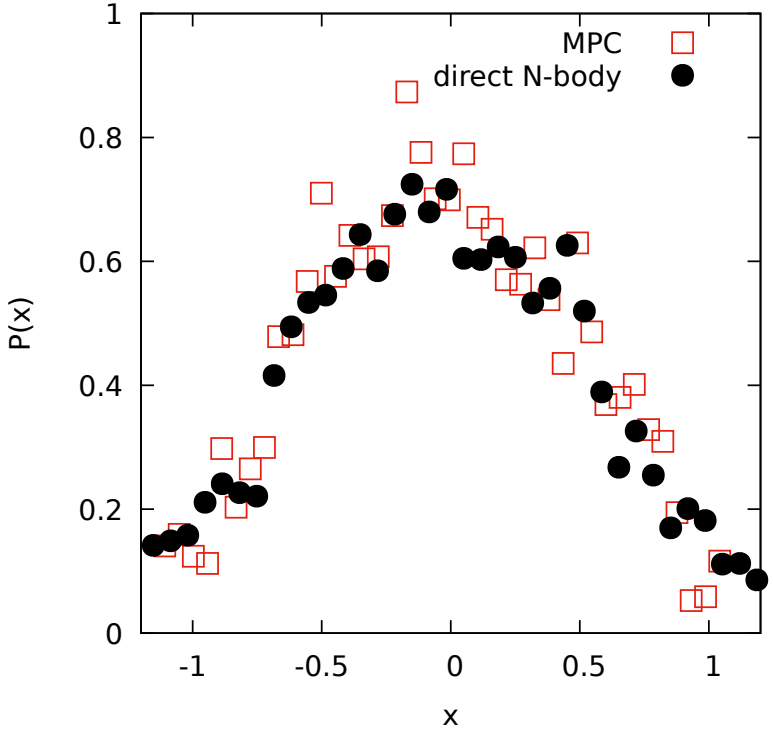


Comparison with direct N-BODY: Orbital structure



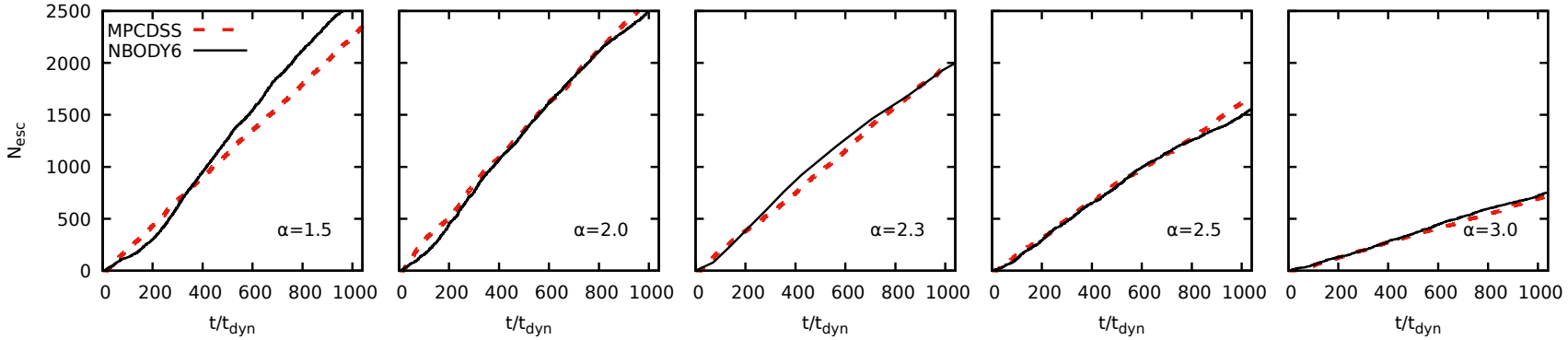
Comparison with direct N-BODY: Orbital structure





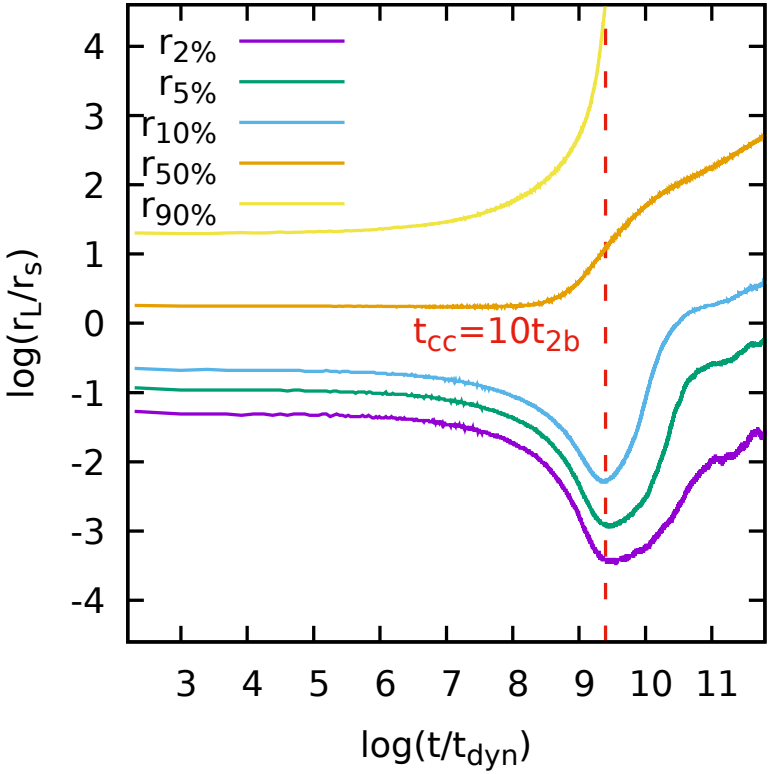
On average the orbital structure is not altered. For a 30K particle system $100t_{2b}$ are simulated in a matter of hours while for a single core run with a direct code it takes days.

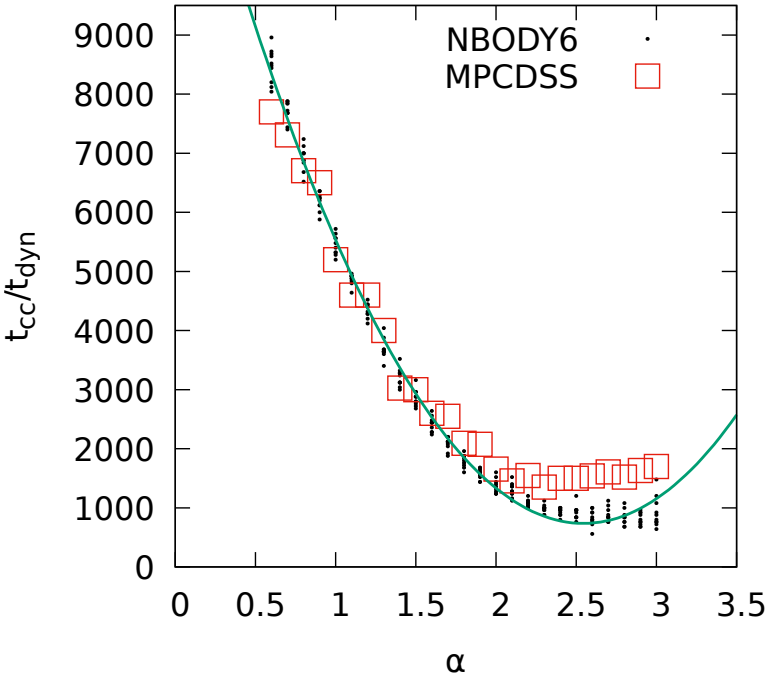
Comparison with direct N-BODY: Escapers



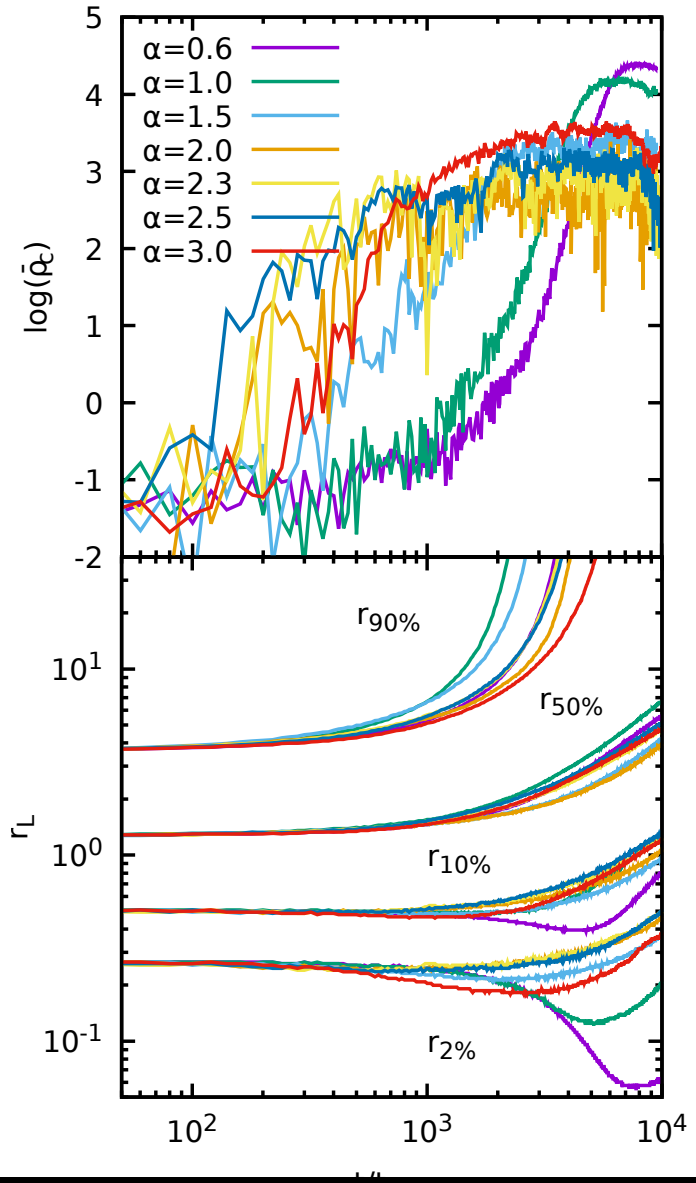
We retrive a linear trend in the fraction of escapers in systems with mass spectrum $f(m) \propto C/m^{\alpha}$.

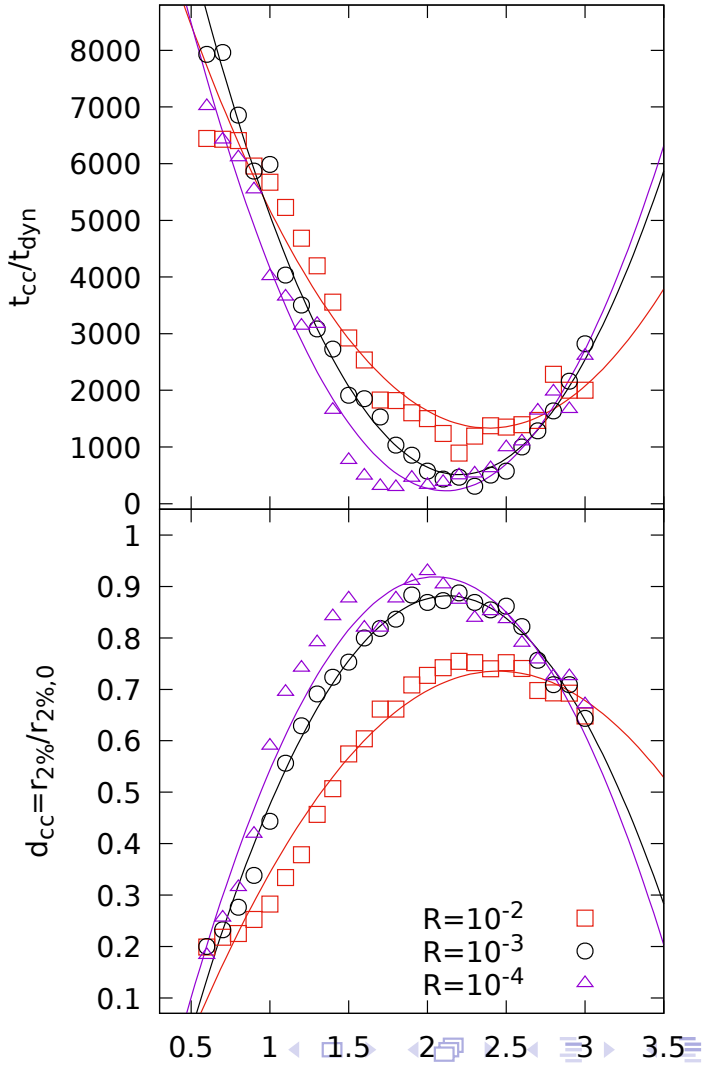
Application to GCs with core collapse: Time and depth of CC



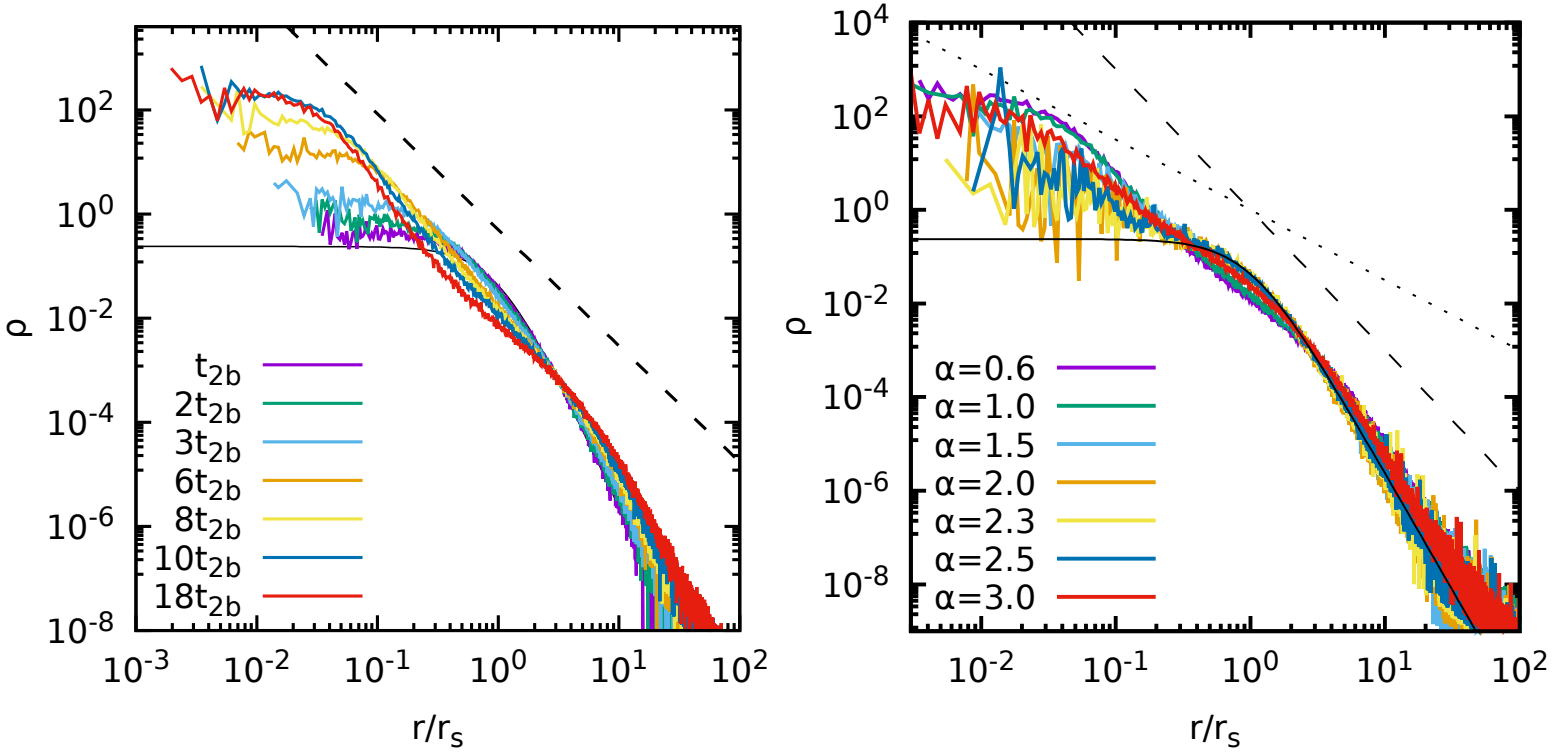


Application to GCs with core collapse: Time and depth of CC



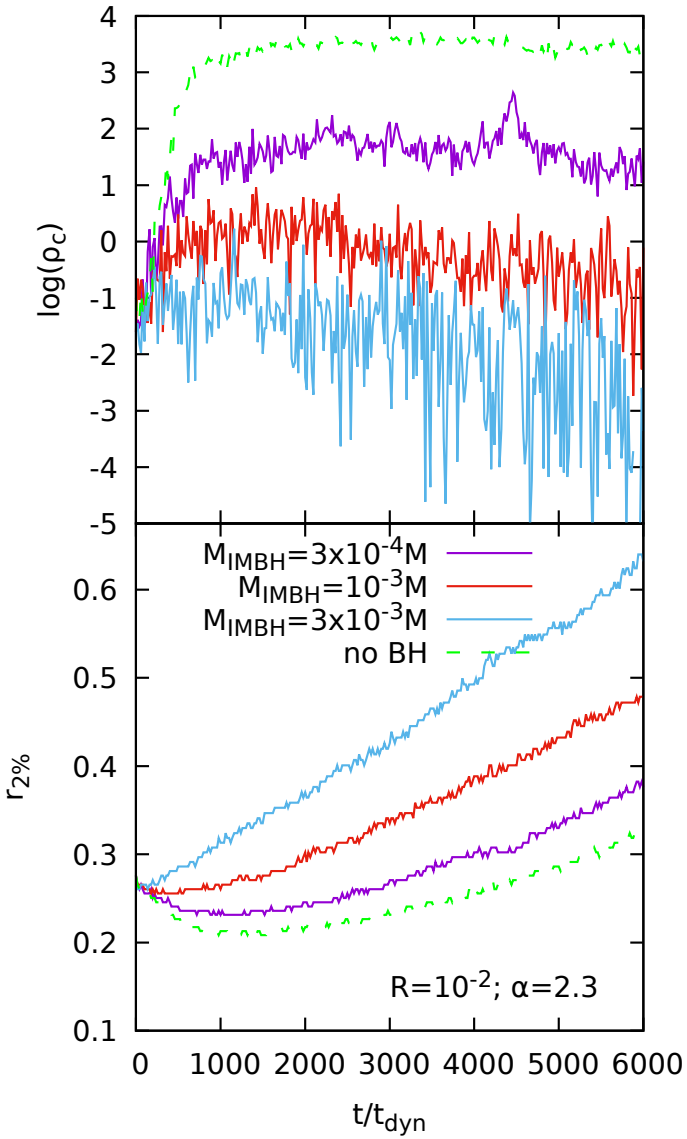


Application to GCs with core collapse: Density

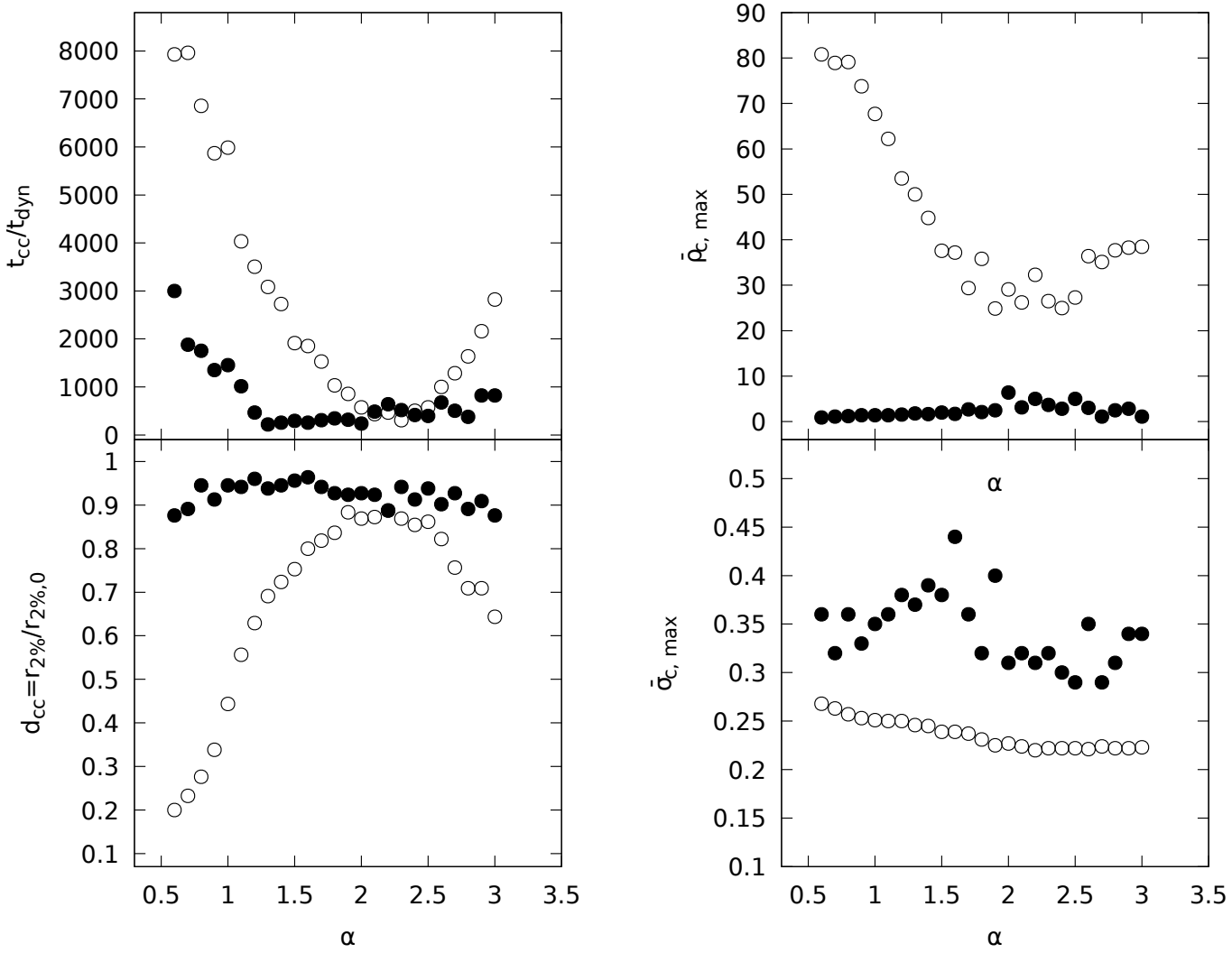


Density profile $\rho(r) \propto \rho_0/r^{-2.23}$ appears after core collapse as in $N-$body simulations by Kupper et al. (2008) and MonteCarlo by Hurley & Shara (2012) and Joshi et al. (2000).

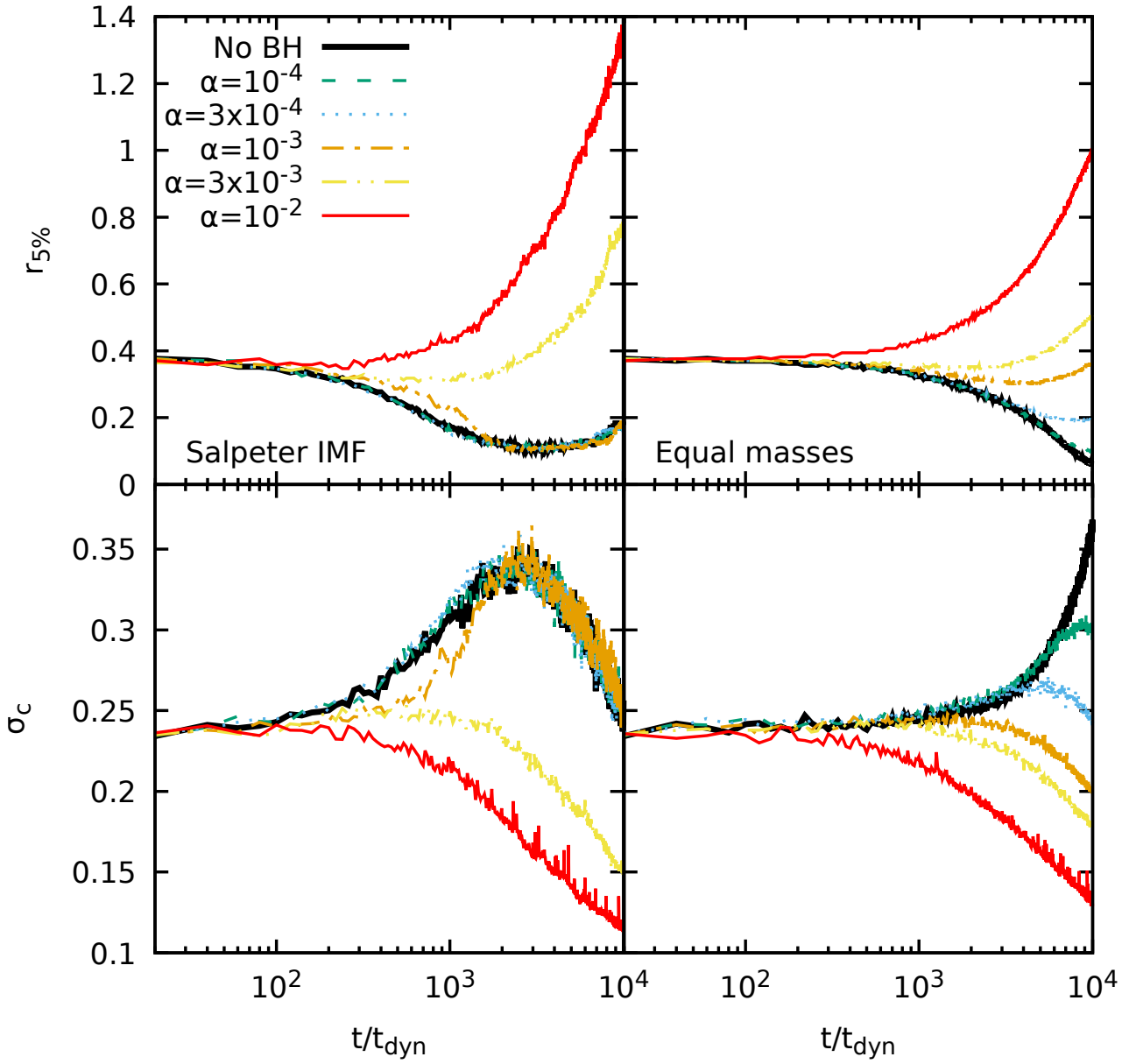
Application to IMBHs



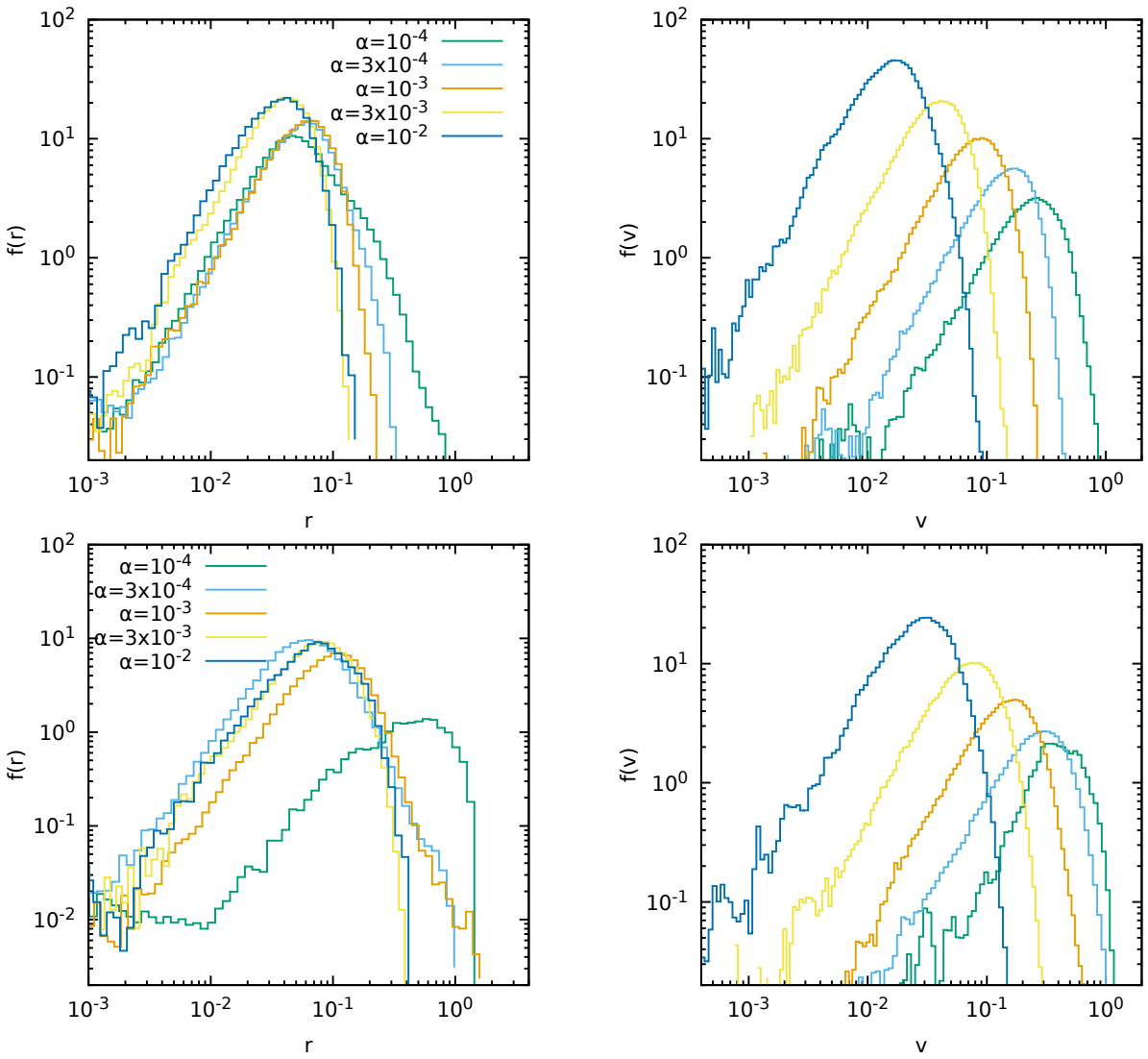
Application to IMBHs



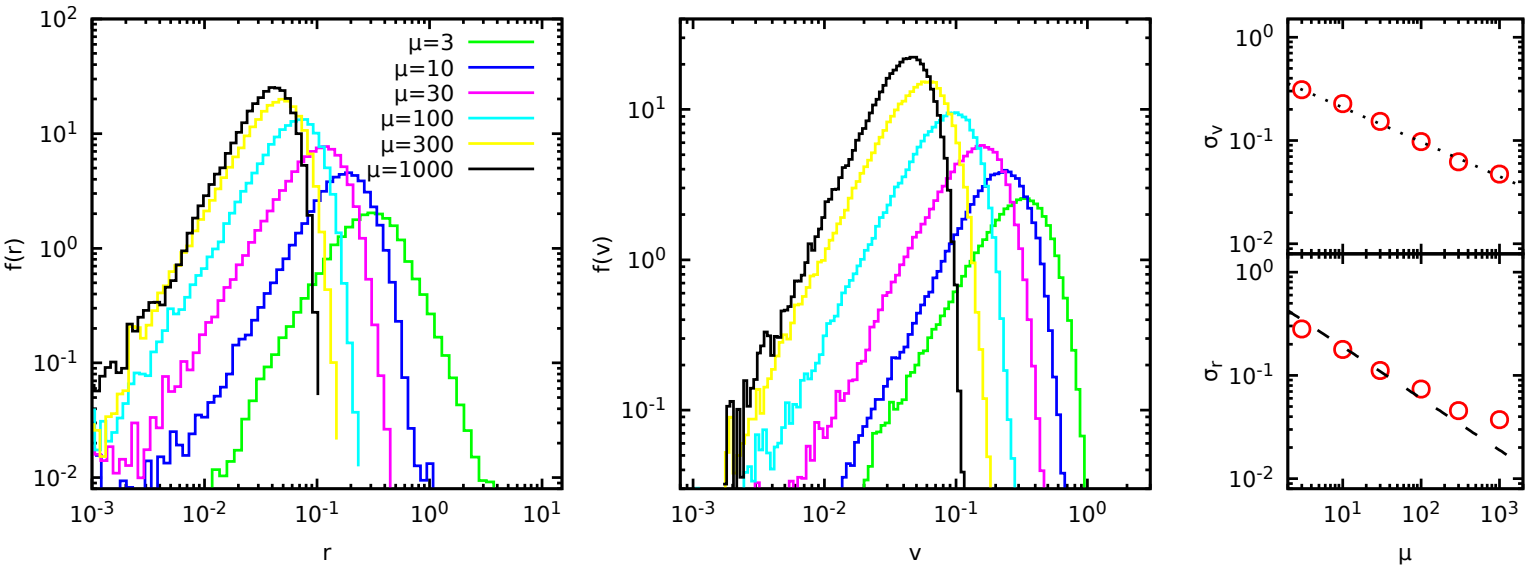
Application to IMBHs



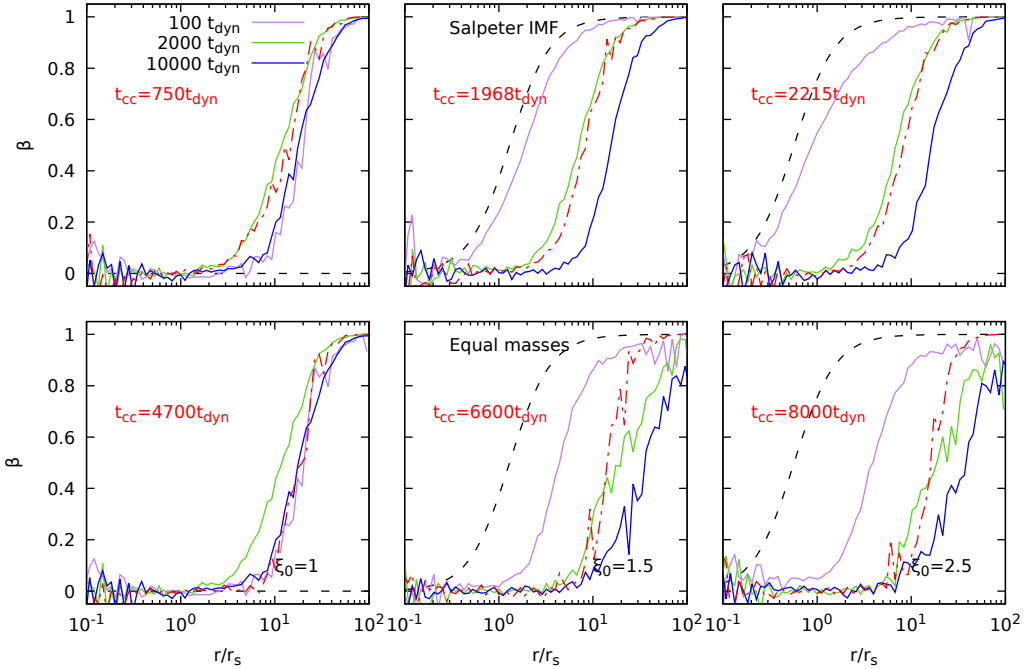
Application to IMBHs



Application to IMBHs



Application to IMBHs



Outlook and perspectives

- We have a tool that is fast and able to treat statistically fairly large collisional systems
- Reliable for treating averaged properties of orbits, even if there is a large degree of approximation
- Include more accurate regularization to treat interaction with compact objects in GCs.
- Include the effects of stellar evolution
- More realistic models with core collapse and binaries

THANK YOU FOR THE ATTENTION!

References

This presentation was based on

1. P. Di Cintio & L. Casetti MNRAS 494, 1027 (2020)
2. P. Di Cintio & L. Casetti IAUS Proceedings 351, 426 (2020)
3. P. Di Cintio, L. Ciotti & C. Nipoti IAUS Proceedings 351, 93 (2020)
4. P. Di Cintio & L. Casetti MNRAS 489, 5876 (2019)

References

1. Di Cintio, P.; Livi, R.; Lepri, S.; Ciraolo, G. PhysRevE 95, 043203 (2017)
2. Di Cintio, P. Gupta S. & Casetti L. MNRAS 475, 1137 (2018)
3. Di Cintio, P.; Pasquato M.; Kim, H.; Yoon, S-J.; A& A 649, A24 (2021)
4. Di Cintio, P.; Pasquato M.; Simon-Petit, A.; Yoon, S-J.; A& A 659, A19 (2022)
5. Di Cintio, P.; Pasquato M.; Trani, A.A.; Barbieri, L.; di Carlo U.N. in preparation (2022).